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ABSTRACT

This publication is designed to aid schools in planning a course of study for students in Course III Mathematics in New York State schools. This guide continues the developmental program begun in Course I. It is intended to serve as the Regents syllabus for the third year of the academic sequence for a New York State Regents diploma. This document includes: (1) an overview of the updates provided by this guide and the scope of its coverage; (2) course objectives; (3) a discussion of the role of problem solving in mathematics education; (4) a course outline including details of various content to be covered and teaching ideas; and (5) a list of resources for teachers. A "Statement of Regents Goals for Elementary and Secondary Students--1984," and a guide to modifications which can be made for handicapped students are appended. (CW)

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Three-Year Sequence for High School Mathematics

Course III

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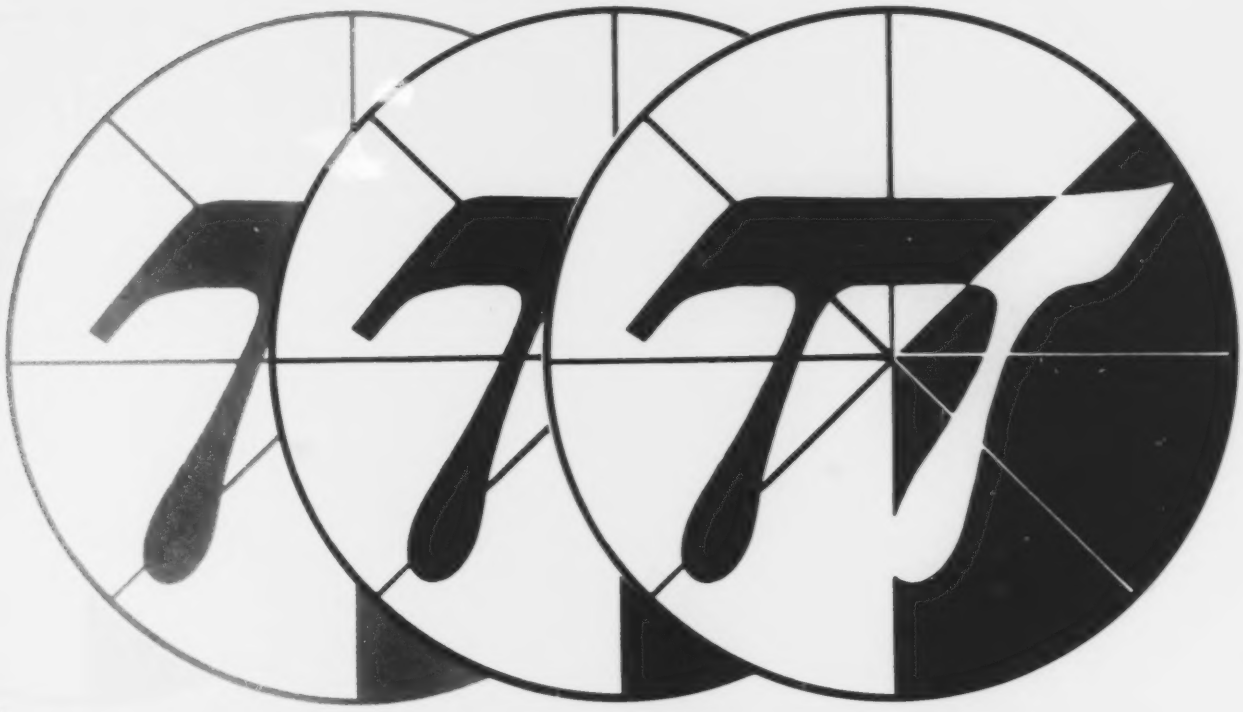

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The University of the State of New York
The State Education Department
Bureau of Curriculum Development
Albany, New York 12234

THE STATE EDUCATION DEPARTMENT / THE UNIVERSITY OF THE STATE OF NEW YORK / ALBANY, N.Y. 12234

TO: Persons with Responsibilities for Implementing High School Mathematics Programs

FROM: Edward T. Lalor, Director, Division for Program Development

Charles J. Trupia, Director, Division of Arts and Sciences Instruction

This publication, *Three-Year Sequence for High School Mathematics: Course III*, is designed to aid schools in planning a course of study for students in Course III. This *Syllabus/Guide* continues the developmental program begun in Course I. It is intended to serve as the Regents syllabus for the third year of the academic mathematics sequence for a Regents diploma, as well as the basis for local activities such as:

Development of local courses of study to meet the standards of the *Syllabus/Guide*.

Selection and acquisition of support materials.

Articulation of local offerings from grade to grade or building to building.

Evaluation of student performance.

As is the policy of the State Education Department, this publication will be periodically reviewed. Your comments on it are welcome and should be addressed to:

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New York State Education Department
Albany, New York 12234

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Three-Year Sequence for High School Mathematics

Course III

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Foreword

I Purpose

The purpose of this *Syllabus/Guide* is to help teachers and administrators plan and implement Course III in the State Three-Year Sequence for High School Mathematics for Regents credit. Course III builds upon the State mathematics curriculum outlined in the Department's mathematics *K-6*, *Grade 7*, *Grade 8*, *Course I*, and *Course II* publications.

II Audience

The audience for this publication includes teachers of courses in the Three-Year Sequence for High School Mathematics; high school mathematics chairpersons; coordinators and curriculum developers; and administrators.

III. Development

This *Syllabus/Guide* has been developed by the Department over a period of several years, and reflects contemporary thinking about the mathematics curriculum for the high school.

As such, this document addresses a 1985 recommendation of the College Entrance Examination Board: the high school mathematics curriculum should be organized so that the topics of computing, statistics, algebra, geometry, and functions are integrated into a three-year series of courses. Also, an important feature of this *Syllabus/Guide* is attention to problem solving, per the recommendations of the National Council of Teachers of Mathematics' "Agenda for Action: Recommendations for School Mathematics of the 80's."

The initial step in preparing this *Syllabus/Guide* to implement the above recommendations was the appointment, in May 1984, of an ad hoc committee by the Bureau of Mathematics Education. The charge of this committee was to advise the Department regarding direction for the mathematics curriculum in the high school. The committee, which met in June 1984, consisted of the following:

Steven Adrian, Mahopac High School

Janet Burt, Clinton High School

Walter Callahan, Shenendehowa Middle School, Elnora; President, Association of Mathematics Teachers of New York State, 1983-84

Bernard Fleishman, Professor of Mathematics, Rensselaer Polytechnic Institute, Troy

Joel Friedberg, Herbert H. Lehman High School, New York City; President, Mathematics Chairmen's Association of New York City, 1984-86

Jack Graver, Professor of Mathematics, Syracuse University

Meyer Jordan, Professor of Mathematics, Brooklyn College

Irwin Kaufman, Director of Mathematics, New York City

Edward Keenan, The Wheatley School, Old Westbury

Edith Luchins, Professor of Mathematics, Rensselaer Polytechnic Institute, Troy

Timothy McMullin, Vestal Senior High School

Lester Rubenfeld, Professor of Mathematics, Rensselaer Polytechnic Institute, Troy

Sister Ann Xavier, Nazareth Academy, Rochester

In June 1984, the State Education Department announced that the integrated three-year sequence would become the official Statewide program for Regents high school mathematics beginning in September 1987. A Revision Advisory Committee was appointed in July 1984. This committee has since met regularly to review draft materials, and to provide input to the Department. It has included the following:

Janet Burt, Clinton High School

David Call, Fort Ann Central School

Roberta Eisenberg, Beach Channel High School, New York City

Pierre Fabinski, Christopher Columbus High School, New York City

Grahme Fischer, Technical Problem Solvers, Hauppauge

Anne Flansburg, Lafayette High School, Buffalo

Bernard Fleishman, Rensselaer Polytechnic Institute, Troy

Robert Frankel, Tottenville High School, New York City; President, Mathematics Chairmen's Association of New York City, 1986-88

Joel Friedberg, Herbert H. Lehman High School, New York City; President, Mathematics Chairmen's Association of New York City, 1984-86

Jack Graver, Syracuse University

Richard Klutch, Hunter College High School, New York City

Sharon Lanza, Grover Cleveland High School, Buffalo

Edith Luchins, Rensselaer Polytechnic Institute, Troy

Shelley Mora, Clara Barton High School, New York City

Fred Pedersen, Herricks High School, New Hyde Park

Lester Rubenfeld, Rensselaer Polytechnic Institute, Troy

George Stevens, Lansing High School

Donald Thorpe, Vestal Senior High School

Sister Ann Xavier, Nazareth Academy, Rochester

Using the recommendations of the Advisory Committee, Sister Ann Xavier, Janet Burt, and Richard Klutch organized and wrote a *Course III* draft *Syllabus/Guide*. Eleven schools/school districts fieldtested it during the 1987-88 school year:

Buffalo City School District

Colonie Central High School, Albany

Dobbs Ferry High School

The Wheatley School, East Williston

Fort Ann Central School

Herricks High School

Mahopac High School

New York City

- Christopher Columbus High School
- Clara Barton High School
- Herbert H. Lehman High School
- Tottenville High School

Nottingham High School, Syracuse

Rochester City School District

Vestal Senior High School

This *Syllabus/Guide* was prepared under the direction of Ben Lindeman and Lee Negus, Associates in the Bureau of Mathematics Education, together with Robert Carruthers, Associate in the Bureau of Curriculum Development. Staff from the Office for the Education of Children with Handicapping Conditions reviewed the manuscript in the light of the needs of students with handicapping conditions. Michael Moon, Associate in the Division of Civil Rights and Intercultural Relations, reviewed the manuscript for equity concerns. Marjorie Edelstein, Christopher Columbus High School, New York City, reviewed the manuscript for environmental concerns. Mary Theresa Southworth, Consultant for the Bureau of Curriculum Development, assisted in the revising and editing of the final manuscript.

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Introduction

Overview. This *Syllabus/Guide* presents an outline and some teaching notes for Course III of the Three-Year Sequence for High School Mathematics program of New York State. First, the topics included in the program are spiraled and sequenced throughout the three years so that the students review and build on previous learning. Second, the branches of mathematics, — including the traditional topics of algebra, geometry, and trigonometry, and the additional topics of logic, probability, statistics, and transformation geometry, — are integrated. Suggestions for such integration are included throughout the *Syllabus/Guide*. Although the topics to be taught are presented here in a traditional unit format for the convenience of teachers and supervisors, teachers should integrate these topics in their lessons when possible. This integration, however, should be a natural, not a forced, one which benefits the instructional process and the learner outcomes.

Use of the Syllabus/Guide. It is expected that the material presented here can be completed by most classes in one school year, usually the 11th grade. Some students will complete the course in less than one year; others will require more than one year. Likewise, although the course is written with the 11th grade student in mind, some students may begin the material either in an earlier or a later grade. It is expected that individual teachers will consider their own students in making judgments concerning time allotments, aspects of the material to be stressed, the degree of rigor employed with any topic, and extra material to be presented. It should be noted, as in Courses I and II, that teachers need not present the topics in the order given in the outline.

Changes from the 1978 edition. Changes in the Course III syllabus include (1) more specificity for topics, (2) an attempt to infuse problem solving throughout, (3) the transfer of circles from Course I to Course III, (4) the recognition of the equations of the conic sections, (5) inverse variation in conjunction with the hyperbola, and (6) the definition of a field. To make room for these topics in Course III, a greater number of algebraic skills is developed in Courses I and II, transformations start in Course I, and right triangle trigonometry is included in Course II.

MAJOR AREAS

The units of Course III conclude the Regents sequence mathematics program of New York State. The following is a brief statement for each major area of the curriculum.

ALGEBRA: The algebra of Courses I and II emphasizes the skills associated with the solution of linear and quadratic equations and inequalities. Course III extends algebraic skills in the areas of fractions, radical expressions, and equations and inequalities dealing with absolute value. A new area of study in this course is that of the complex numbers. Concepts and arithmetic operations with complex numbers are followed by a review and extended study of quadratic equations - this time allowing such equations to have solutions in the complex number system. A major objective of Course III is to increase the students' sophistication in using algebra as a tool in solving problems.

RELATIONS AND FUNCTIONS: Much of the work of Course III and other mathematics courses to follow in high school depends heavily on an understanding of functions. This unit of Course III provides a study of relations and functions and the symbols and terminology, composition, and inverses of functions. The student is introduced to the exponential function and its inverse, the logarithm function. Emphasis is placed on the properties of the logarithm as a function, rather than its computational aspects. In a later unit, the study of trigonometry is approached from the viewpoint of function.

TRANSFORMATIONS: The revisions of Courses I, II, and III have taken elements of the material on transformation geometry from the original Course III and introduced them in the two preceding courses. Course I addresses, intuitively, elementary notions associated with reflections, translations, rotations, and dilations. Course II looks at these transforma-

tions from the viewpoint of coordinate geometry. In Course III, the emphasis is on the composition of two or more transformations and on the application of transformations to the study of other areas of mathematics.

TRIGONOMETRY: The six trigonometric functions are studied, using many of the ideas from the transformations unit. Identities and conditional equations are presented, integrating the algebraic skills of working with fractions and solving equations with the notions of the trigonometric functions. The general solution of the triangle using the Law of Sines and the Law of Cosines again integrates the students' algebraic skills in problem-solving situations.

GEOMETRY: The study of the circle completes the geometry strand of the sequential mathematics program. Theorems are presented concerning angle measurement, arc measurement and lengths of line segments, including tangents, secants, and chords.

PROBABILITY: Course III includes a review of combinatorics studied in Courses I and II. The probability associated with Bernoulli experiments is presented, along with a study of the Binomial Theorem.

STATISTICS: Following a review of the measures of central tendency, the measures of dispersion are studied, including the range, the variance, and the standard deviation. A brief introduction is made to the normal distribution and its connection with the standard deviation.

OTHER CONCERNS

For Course III, in addition to the above content, the items listed below should be considered in the implementation of the syllabus.

ESTIMATION AND APPROXIMATION: Estimation and approximation are important strategies used in elementary and junior high schools and should be continued in Courses I, II and III where appropriate. Estimation is often useful in areas such as probability and statistics, and should be systematically used when solving problems. Estimation, where appropriate, should be one of the steps in the problem-solving model. The students will not use estimation unless they find it useful; therefore, the teacher's role in teaching estimation should be one of modeling and motivating.

COMPUTERS AND CALCULATORS: Course III, like Courses I and II, has no formal units on the computer or the calculator. Nonetheless, the use of calculators and computers is suggested in those areas where such devices can contribute to mathematical understanding, increase the students' participation and enjoyment, or broaden the area from which applications of mathematical ideas can be selected. Individual teachers may wish to give lesser or greater emphasis to the role which these modern tools can play in the study of mathematics.

PROBLEM-SOLVING: Problem solving is a major focus of mathematics education. The teacher should make every effort to use problems in daily teaching. In Course III, problems should make applications of algebraic skills and techniques, geometric concepts, and trigonometric concepts to all areas of the syllabus. Both routine and nonroutine problems should be presented, and the teacher should use problems that integrate topics wherever appropriate.

INTERDISCIPLINARY ACTIVITIES: The Regents Action Plan recommends the use of special interdisciplinary activities to show the students that the various subject areas are interrelated. Science, business, the social sciences (including economics), and technology education make extensive application of mathematical skills and concepts. Problems/situations from these areas will show the students the value of mathematics as a tool used in many human endeavors.

Music and art may also be used as vehicles to show the importance of mathematics in the arts. Much music theory is based upon basic principles of mathematics. Symmetry, tessellations, and form play a significant role in various forms of art.

The role of pure mathematics cannot be overlooked when examining the interdisciplinary aspects of mathematics. Major scientific theories of the 19th and 20th centuries were able to be developed because the mathematics for them had been created before anyone saw their possible application to science. For example, Einstein's general theory of relativity would not have been possible without the geometry of Riemann, invented in 1854, or without the theory of invariance developed by Cayley, Sylvester, and other 19th century mathematicians. Many such examples could be given, to show, as Gauss said, that "Mathematics is Queen of the Sciences."

AN APPROACH: It is hoped that this program will give the students a useful and enjoyable experience with mathematics which will encourage them to continue its study. It is suggested that the teacher give priority to a classroom atmosphere where there is ample time for student investigation, discovery, and discussion rather than to the task of "covering the material." Given sufficient time for the various possibilities these topics have to offer, the students will participate in their high school mathematics with a fuller appreciation

tion of the fact that mathematics is a broader and more exciting field of study than many students perceive it to be.

It is our belief that the program outlined in this mathematics *Syllabus/Guide* reflects what is happening in mathematics today. It is our hope that the students who participate in it will be more likely to accept mathematics as a relevant part of their total experience and to want to continue with it, for their own benefit and for the benefit of all of society.

Goals and Learner Outcomes

The Statement of Regents Goals for Elementary and Secondary Students - 1984 includes the following:

- Each student will master computation skills as a foundation to:
  - think logically and creatively
  - apply reasoning skills to issues and problems
  - perform basic mathematical calculations
  - determine what information is needed for particular purposes and be able to acquire, organize, and use that information for those purposes.
- Each student will learn methods of inquiry and knowledge gained through mathematics, and use these methods and knowledge in interdisciplinary applications.

These Regents Goals act as a cornerstone for the State K-6 and grades 7-8 mathematics syllabi. The Course I, II, and III syllabi build upon and extend this foundation.

The College Board in "Academic Preparation for College" states that all college entrants will need to achieve proficiency in the following:

- the ability to apply mathematical techniques in the solution of real-life problems and to recognize when to apply those techniques
- familiarity with the language, notation, and deductive nature of mathematics, and the ability to express quantitative ideas with precision
- the ability to use computers and calculators
- familiarity with the basic concepts of statistics and statistical reasoning
- knowledge in considerable depth and detail of algebra, geometry, and functions

It is the goal of the syllabi for Courses I, II, and III to provide the students with a mathematical background which will be sufficient to meet their needs now and as they move into the 21st century.

The cultural aspects of mathematics should be made evident to learners so they can look upon it as a human invention important in itself—perhaps the greatest intellectual invention of the human mind. Mathematics has logic, abstraction, paradoxes, and patterns. These attributes should not be overlooked in mathematics instruction.

Mathematics can also be useful for dealing with real problems. These should be an integral part of the students' high school mathematics program. Additionally, effective mathematics programs should help the students to develop an appreciation of mathematical ideas and thoughts, and should cultivate a trust and belief in results based upon mathematics. They should provide for the:

- mathematical competence necessary to function and live independently in society
- mathematical competence in areas essential to function in the work an individual does or aspires to do
- mathematical competence needed to engage in hobbies, games, and leisure time pursuits and to add enjoyment to life.

Mathematics should benefit the students in three major goal areas:

- affective goals–including the appreciation of the structure of mathematics as a liberal art, and an understanding of the role that mathematics has played as a part of our cultural heritage
- content goals–including knowledge, skills, and the language of mathematics
- process goals–including interpretation, application, deduction, and the transfer of mathematical concepts and processes.

General cognitive learner outcomes for Course III include:

- an understanding of the complex number system, including the properties necessary for a mathematical field
- the mastery of algebraic techniques for their unify-

ing value in mathematical thinking and for their application to the real world

- an extension of the solution of the quadratic equation to include equations having imaginary (complex) roots
- an understanding of the concept of functions and their inverses
- the application of geometric transformations to functions and graphing
- an extension of the work with exponents to include the exponential and logarithmic functions
- an extension of geometric proof, including theorems related to angle measure and lengths of segments in a circle
- the basic ideas of trigonometric functions and their graphs, trigonometric identities and equations, and trigonometric formulas
- the ability to use trigonometric relationships to solve triangles and to find area
- the application of the technique of probability to solve problems associated with the Bernoulli experiment
- an introduction to the Binomial Theorem
- an extension of the measures of dispersion, to include standard deviation
- the application of the concepts of statistics in the study of the bell-shaped curve
- the development of problem-solving strategies to solve a wide range of problems involving mathematical applications.

Specific content for Course III is listed in the Course III Outline (pages 11 to 14) of this *Syllabus/Guide* and is described more fully in the sections that follow that Outline. Instruction should be designed to integrate the topics as much as possible, emphasizing the relations between the various segments of the curriculum. Problem solving should be infused throughout the course of study, with concepts and techniques from the various strands introduced as they contribute to an understanding and solution of a problem.

The Role of Problem Solving

Problem solving should be an integral part of the sequential mathematics curriculum. The teacher should continually try to incorporate problem solving into the learning activities for each unit. Problem solving should go beyond the topics of the syllabus, to stimulate discussion, to encourage creativity, and to build a classroom atmosphere in which the students are encouraged to trust their abilities to think and to confront an unfamiliar task.

Problem solving in the classroom serves several purposes. A problem which uses the concepts and skills of a new unit may be presented to the students at the beginning of the unit to show them that they need additional skills to solve the problem. This may motivate them to learn the material in the unit. Business, industry, technology, and science provide realistic problems that generally interest the students and show real-world applications of mathematics. Finally, problem solving provides a vehicle to use estimation and to develop thinking skills as well as to review and reinforce previously presented concepts and skills.

The students should make a habit of estimating, where appropriate, the answer to a problem before solving it. If the solution is not "in the ballpark" of the estimated answer, the student should be aware to check his/her answer or estimate. The model given below includes estimation as one of the steps in solving a problem.

PROBLEM-SOLVING MODEL:

1. Read and understand the problem.
2. Organize the information—diagrams, lists, etc.
3. Devise a plan for solving the problem.
4. Where appropriate, estimate the answer.
5. Carry out the plan.
6. Check the solution.

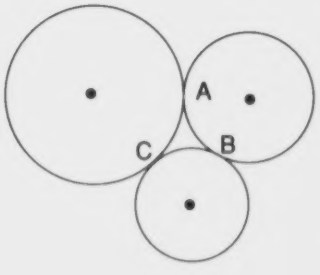
Course III is the third year in the sequential program. The students have been exposed to the spirit of the problem-solving approach described here for the two previous years, and before that, in the seventh and eighth grades. The teacher should be familiar with the problem-solving activities of these courses as an aid in considering directions that problem-solving activities may take during Course III. As the culminating course in the three-year sequence, Course III presents the most varied, challenging, and rewarding opportunities for bringing together, on a more mature level, the various techniques the students (and the teacher) have learned during previous study of mathematics.

In Course I, the emphasis was on using newly-learned elementary ideas about algebra, geometry, and other topics in the curriculum. Course II not only brought with it another year of maturity, but also increased mathematical skills, especially in logic, algebra, and more formal geometry. Course III adds a broader understanding to trigonometry, more sophisticated ideas about relations and functions, a variety of approaches to geometry, and the introduction of a number system beyond the reals, the latter in itself an elegant and efficient construct designed to help probe previously uncharted areas.

Below are some examples of problem-solving activities. These are included to give the teacher not only an idea of the spirit of the problem-solving goal here discussed, but also some useful examples for integrating problem solving throughout Course III.

The first two problems might be called, "Ah-ha, I have it!" problems. (A flash of insight often helps!)

1. Three circles are mutually externally tangent at points A, B, and C. Find the sum of the degree measures of $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, and $\overset{\frown}{CA}$.



The students find that looking only at the arcs does not provide an answer. The light dawns when a student finds that by constructing the triangle with the three centers as its vertices, each angle subtends one of the desired arcs and "Ah-ha, I have it!"

2. An urn contains 5 red and 12 blue marbles. While a marble is being drawn at random from the urn, a fair coin, with one side red and the other side blue, is tossed. Find the probability that the two selections are the same color. The student who uses the formula finds:

   P(R and R) + P(B and B) =

   $$\frac{5}{17} \times \frac{1}{2} + \frac{12}{17} \times \frac{1}{2} = \frac{17}{34} = \frac{1}{2}$$

   However, the student with insight may say, "Ah-ha, I have it!" No matter what color marble is selected from the urn, the coin's color has a 1/2 probability of matching it.

Problems 3-8 make use of the content of the Course III syllabus. However, solutions are easier with pictures, diagrams, tables, charts, a little insight, etc.

3. When a piece of paper is folded, its thickness doubles. Assuming a piece of paper is approximately 0.002 inches thick, how many folds will it take for the thickness of the paper to be at least one foot?

   After the calcuation of a few thicknesses, a function for the thickness of the paper can be written as follows:

# of folds	thickness
0	0.002
1	0.004
2	0.008
3	0.016

   $T(f) = (0.002) \cdot 2^f$

   Thus we want the function $(0.002) \cdot 2^f \geq 12$.

   The students can use calculators, or logarithms, to find the answer to be f = 13.

4. If one side of the square in the diagram measures 12, what is the area of the shaded portion?

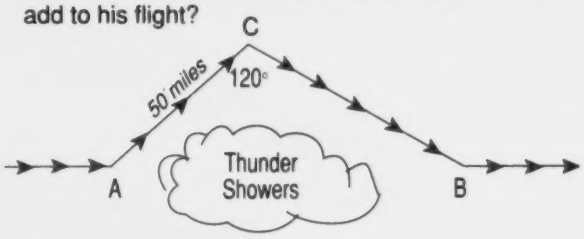
   The symmetry of the figure suggests that we may work with one half of one leaf of the shaded area. The area of this half leaf is the area of a quarter of a circle of radius 6 minus the area of an isosceles right triangle with the length of a leg equal to 6.

   $$\frac{1}{4}(\pi(6)^2) - \frac{1}{2}(6)(6) = 9\pi - 18$$

   The area of the shaded figure is 8 $(9\pi - 18)$ or $72\pi - 144$.

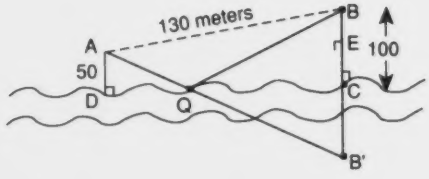
5. There are many opportunities in the study of triangle trigonometry to pose problems requiring an analysis and use of formulas and concepts from the study of trigonometry.

   A pilot, flying from New York to California, encounters an area of severe thunderstorm activity. He changes his flight pattern by going off course for 50 miles at an angle of 35°. Then he turns back toward his original course at an angle of 120° and returns to his original flight plan. How many extra miles did this diversion to avoid the thunderstorm add to his flight?



   The students will need to use the Law of Sines to find distances BC and AB to determine the extra mileage.

6. Tom is at point A, 50 meters from the bank of a river. He needs to go to the river, get a pail of water, and take it to point B, which is 100 meters from the river. Points A and B are 130 m apart. To what point on the river bank should he go so that the distance he travels from A to the river to B will be a minimum?



   This problem can be easily solved if the student uses her/his knowledge of geometry and transformations to analyze the situation.

   Of all the possible paths which Tom may take, the path which leads to Q, a point on the river which is also on the straight line joining point A to the reflection of point B, must be the shortest path. It can be shown that BQ = B' Q, and AQ + QB = AQ + QB'. Since the shortest distance between two points is a straight line, AB, which is equal to AQ + QB , is the shortest distance.

   Once the position of Q is established, the student can prove Δ ADQ ~ Δ BCQ and use a proportion to solve for the actual distance. First, however, the distance DC must be determined. This can be shown by use of a right triangle, Δ ABE, to be 120 meters.

The proportion, $\frac{x}{120-x} = \frac{50}{100}$, can be solved to find the distance x = 40. Thus Tom should go to point Q, which is 40 meters from D, get the water, and proceed to B.

7. If the product of two numbers is 20, and the sum of the squares of the two numbers is 49, find the square of the difference of the two numbers.

   Expressed algebraically, xy = 20 and $x^2 + y^2 = 49$. The problem asks for the value of $(x-y)^2$. The student could solve these two equations simultaneously to find the values of x and y. However, a much easier approach is to express the square of the difference of x and y as follows:

   $(x-y)^2 = x^2 - 2xy + y^2$

   Then, from the given information, xy can be replaced by 20, and $x^2 + y^2$ can be replaced by 49. Hence the value of $(x-y)^2$ is 9.

   A similar problem: The sum of two numbers is 16, and their product is 67. Find the sum of the squares of the two numbers.

   Analysis: xy = 67 and x + y = 16. We are to find $x^2 + y^2$.

   $(x+y)^2 = x^2 + 2xy + y^2$

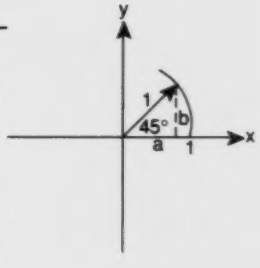
   $(16)^2 = x^2 + 2(67) + y^2$

   $256 = x^2 + 134 + y^2$

   $122 = x^2 + y^2$

8. Find $\sqrt{i}$ in a + bi form. To solve this problem, we will make some assumptions, and may examine some ideas that could be a precursor of DeMoivre's Theorem in Math 12. Using the graphic representation of a complex number, consider i as the image of 1 under a rotation of 90° about the origin. $i^2 = -1$, which is the image of 1 under a rotation of 180°. 180° = 2 (90°). $(-1)^2 = 1$ and 360° = 2 (180°). $(-i)^2 = -1$ and 540° = 2 (270°), etc. It appears that when we double the angle of rotation from 1, we square the image number. Maybe taking half of the angle of rotation will give us the square root of the image number.

   Since i has a rotation angle of 90°, maybe $\sqrt{i}$ has a rotation angle of 45°. We will investigate the image of 1 under a rotation of 45° as a possible value for $\sqrt{i}$. Since an isosceles right Δ with hypotenuse of 1 is formed,



   a = b and $a^2 + b^2 = 1$

   $2a^2 = 1$

   $a = \frac{\sqrt{2}}{2}$ and $b = \frac{\sqrt{2}}{2}$.

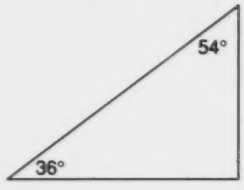
   $\therefore \sqrt{i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

   This result may be verified by showing that $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)^2 = i$.

The following problem, although far from simple, has some of the features which the teacher should look for when presenting new situations to a class. It can be looked at from many different directions. It is clearly non-routine. And it integrates several of the strands (geometry, algebra, trigonometry) the students have encountered in the sequential program.

9. Find the *exact value* of cos 36°.

   It should be made clear that an approximate value, and/or the value from a trig table, is not what is being sought. The students might examine without success situations that include an angle whose measure is 36°, such as the central angle of a regular decagon, or the bisected central angle of a regular pentagon. However, if a right triangle is drawn with one angle measuring 36°, the other acute angle has a measure of 54°. If a relationship can be shown, some of the trig formulas may be used.



   After some experimentation, the students may note that 36:54 = 2:3. Therefore the angle measures may be designated 2θ and 3θ and:

   $\sin 2\theta = \cos 3\theta$

   Using appropriate formulas and the fact that $\cos 3\theta = \cos(2\theta + \theta)$, we obtain:

   $2 \sin\theta \cos\theta = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

   $= \cos 2\theta \cos\theta - 2 \sin\theta \cos\theta \sin\theta$

   $2 \sin\theta \cos\theta = \cos 2\theta \cos\theta - 2 \sin^2\theta \cos\theta$

   Dividing both sides by $\cos\theta$ ($\cos\theta \neq 0$)

   $2 \sin\theta = \cos 2\theta - 2 \sin^2\theta$.

   Substitute $\cos 2\theta = 1 - 2 \sin^2\theta$

   $2 \sin\theta = 1 - 4 \sin^2\theta$ or

   $4 \sin^2\theta + 2 \sin\theta - 1 = \theta$

   and $\sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$

   choosing the positive value: $\sin\theta = \frac{-1+\sqrt{5}}{4}$, i.e.

$$\sin 18^\circ = \frac{-1 + \sqrt{5}}{4}$$

Finally, using $\cos 2\theta = 1 - 2\sin^2\theta$

$$\cos 36^\circ = \frac{1 + \sqrt{5}}{4}$$

Note that, in addition to the trig formulas, this problem provides effective algebraic review.

The ability to confront nonroutine tasks — that is, to deal successfully with those tasks that require the selection of a procedure that is not immediately obvious, — is a desirable outcome of education in mathematics. This skill does not develop automatically with the mastery of computational skills and techniques; it must be developed through practice, through the sharing of insights, and through the development of the habits of questioning and curiosity.

As teachers of mathematics, we see our subject as a logical and coherent set of concepts that develop naturally one from another. Many of our students, however, do not see it that way because, often, in the interest of "covering the course" or "completing the work" of a particular year, the teacher presents mathematics as a set of rules to be learned rather than as ideas that unfold or emerge from the learner's ability to see and appreciate relationships. Time given to problem solving can be time well spent because the enthusiasm and excitement which can be generated lead to more retentive learning.

The purpose, then, of this introductory section is to encourage the use of nonroutine problems throughout the course in addition to those problems that are part of the content of each unit of the course. When chosen appropriately, classroom activities and applications can serve a multi-purpose role in applying mathematics concepts and skills to important issues. For example, the students can increase their awareness of various handicapping conditions by applying such concepts and skills to the use of wheelchair ramps, medication, adaptive devices, etc.

Problems may serve the following purposes:

1. to motivate the introduction of a new topic or the development of a new skill or technique;
2. to provide practice with a skill that has been or is being developed;
3. to allow the exploration of concepts beyond those suggested by the content of the course;
4. to develop in the individual student the confidence to try various techniques when faced with an unfamiliar problem.

Course III Outline

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2. Field properties

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4. Simplifying radical expressions
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6. Simplifying algebraic fractions
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8. Addition and subtraction of fractions
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   b. sum and product of roots
   c. relationship between the quadratic equation and the quadratic function
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A Note on Time and Sequence

It is difficult to supply a meaningful time frame for an integrated course since it is intended that topics interact with each other in the development of the material. The following chart should be used only as a general frame of reference.

	Topic	*Days*
I.	The Complex Number System	35 – 37
II.	Transformations	10 – 12
III	Relations and Functions	21 – 23
IV.	Circles	10 – 12
V.	Trigonometric Functions and Inverses	25 – 27
VI.	Trigonometric Identities and Equations	10 – 12
VII.	Triangle Trigonometry	10 – 12
VIII.	Probability	8 – 10
IX.	Statistics	8 – 10
	TOTAL	137 –155

Depending on the students' backgrounds, time allotments for one or more of these topics may vary significantly.

The above time suggestion is not intended to imply that these topics should be taught in this order or as separate blocks of material. For example, algebraic techniques will be used throughout the course in the development of other topics.

I.
The Complex Number System

The algebraic work in Courses I and II increased the students' level of understanding and skill in working in the real number system. Considerable attention was given to techniques necessary to work with algebraic fractions, linear equations, and quadratic equations. In Course II, properties of real numbers were included to give some understanding of mathematical structure as related to algebraic skills used.

In Course III, the students will extend their understanding of the real number system and will be introduced to the complex number system. The definition and properties of a field should be used in the development of the complex numbers.

A. REVIEW OF THE REAL NUMBERS

1. Subsets of the Reals

The development of the real number system usually progresses from the natural numbers through the whole numbers, integers, and rationals to the reals. These familiar sets of numbers should be reviewed. The students should be aware that certain sets, for example the whole numbers or the integers, are not closed with respect to one or more of the operations. The whole numbers do not contain additive or multiplicative inverses for each element, and the integers do not contain multiplicative inverses for each element. Thus these sets do not form a field as defined below. The rationals are the first subset of the reals in this progression which meet the criteria of a field.

2. Field Properties

The properties of a field should be presented and discussed.

A field is an abstract mathematical system consisting of a set, together with two operations, usually addition and multiplication, meeting the following requirements:

a. closure with respect to both operations

b. associative with respect to both operations

c. commutative with respect to both operations

d. existence of an identity element for each operation

e. existence of an inverse for each element under both operations (excluding 0 under multiplication)

f. distributive with respect to multiplication over addition.

The set of reals and its usual subsets should be explored to determine which meet the requirements of a field. The students should also discuss the implications which these field properties have for the solutions of equations and inequalities and for the various techniques used with algebraic fractions.

B. REVIEW AND EXTENSION OF ALGEBRAIC TECHNIQUES

The students have had some experience with most of the techniques reviewed in this section. However, mastery may not yet have been attained. The nature of the students' background will determine the amount of time needed for the review of these topics. In most cases, greater depth is expected in Course III.

1. Factoring

The work here is a continuation of previous work with factoring.

The students should recognize that the process of factoring is an application of the distributive property. They should be able to remove a common monomial factor, recognize the difference of two perfect squares, and recognize a perfect square trinomial. Factoring examples should include a combination of these techniques.

Examples:

$4x^2 - 9 = (2x + 3)(2x - 3)$

$6x^3y - 8x^2y^2 = 2x^2y(3x - 4y)$

$6x^3 + 27x^2 - 15x = 3x(2x - 1)(x + 5)$

$x^4 + 2x^2 + 1 = (x^2 + 1)(x^2 + 1)$

2. Laws of Exponents

The following laws of exponents should be reviewed where x is a real number and a and b are integers.

(1) $x^a \cdot x^b = x^{a+b}$

(2) $\frac{x^a}{x^b} = x^{a-b} \quad (x \neq 0)$

(3) $(x^a)^b = x^{ab}$

(4) $(xy)^a = x^a y^a$

3. Integral Exponents

The students should be familiar with positive and zero exponents. The rules of exponents given above may be used to justify the use of negative integers as exponents.

x^n means $\underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}_{\text{n factors, where n is a positive integer.}}$

$x^0 = 1 \quad (x \neq 0)$

$x^{-n} = \frac{1}{x^n}, \quad (x \neq 0)$

These definitions and properties should be used in simplifying expressions with integral exponents.

Examples:

$(4x^2y)^2(3xy^3) = 48x^5y^5$

$\frac{32x^3y^2z^2}{4x^4yz^6} = 8x^{-1}yz^{-2}$ or $\frac{8y}{xz^2}$

If $x = 4$, $3x^0 + x^{-1} = 3(1) + \frac{1}{4} = 3\frac{1}{4}$.

4. *Simplifying Radical Expressions*

The students should be able to simplify expressions containing radicals, and should be able to add, subtract, multiply, and divide radical expressions.

5. *Rationalizing Denominators*

Expressions such as $\frac{1}{\sqrt{2}}$ or $\frac{3}{2+\sqrt{3}}$ can be rewritten with rational denominators by using the multiplication property of 1.

Example 1:

$$\frac{1}{\sqrt{2}} = \frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}}$$

Example 2:

$$\frac{3}{2+\sqrt{3}} = \frac{3}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{6-3\sqrt{3}}{1}$$

6. *Simplifying Algebraic Fractions*

Using factoring and the multiplication property of 1, the students should be able to simplify algebraic fractions.

Example:

$$\frac{5x^2 - 15x}{27x - 3x^3}$$

7. *Multiplication and Division of Fractions*

Again using factoring and the multiplication property of 1, the students should be able to multiply and divide algebraic fractions and to reduce the answers to simplest form.

Example 1:

$$\frac{2x^2 - x - 6}{x^2 - 5x - 6} \cdot \frac{2x^2 - 12x}{2x - x^2}$$

Example 2:

$$\frac{x^4 - 1}{2x^2 + x - 3} \div \frac{x^3 + x}{2x + 3}$$

8. *Addition and Subtraction of Fractions*

The students' previous work with least common denominators should be reviewed and applied to combining algebraic fractions.

Example:

$$\frac{5}{x+1} - \frac{6}{x-1}$$

9. *Simplifying Complex Fractions*

A complex fraction is a fraction whose numerator or denominator (or both) contains one or more fractions. The process of simplifying a complex fraction is an extension of the algebraic techniques which the students have previously used.

A fraction such as $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x^2}}$ may be treated as the division

of $1+\frac{1}{x}$ by $1-\frac{1}{x^2}$.

Another approach is to multiply both the numerator and denominator of the complex fraction by the LCM (least common multiple) of the denominators of the fractions.

The students should recognize that some expressions written with negative exponents are complex fractions.

Example:

$$\frac{1+x^{-1}}{1-x^{-2}} = \frac{1+\frac{1}{x}}{1-\frac{1}{x^2}}$$

C. REVIEW AND EXTENSION OF SOLUTIONS OF EQUATIONS

In Courses I and II, the students solved both linear and quadratic equations. In Course I the quadratics considered were only those whose solutions could be obtained by factoring. In Course II, the quadratic formula was introduced. In Course III, the solution of linear and quadratic equations should be reviewed and extended to include equations with complex roots.

1. Absolute Value

The definition of absolute value should be formally introduced at this time. One definition commonly used is:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using this definition, first degree equations containing the absolute value of an expression can be solved.

Example:

$|2x - 3| = 5$

If $2x - 3 \geq 0$, then $2x - 3 = 5$, or, if $2x - 3 < 0$, then

$2x = 8$ $\quad$ $-(2x - 3) = 5$

$x = 4$ or $-2x + 3 = 5$

$-2x = 2$

$x = -1$

Thus the solution set is $\{-1, 4\}$.

2. Quadratic Equations

The students are expected to be able to use the quadratic formula to solve any equation of the form $ax^2 + bx + c = 0$. Review of the use of this formula with a variety of quadratic equations will prepare the students for solving quadratic equations over the set of complex numbers.

3. Systems of Equations

In Course I, the students solved systems consisting of two linear equations. In

Course II, the students' experience was extended to linear-quadratic systems. The substitution method is generally the one used to solve such a system.

Example: Solve the system:

$y = 4x - x^2$

$y - x = 2$

The students should be reminded that the solution set $\{ (1,3), (2,4) \}$ of this system represents the coordinates of intersection of a straight line and a parabola.

4. Fractional Equations

The students are expected to be able to solve fractional equations. Such equations may lead to the solution of a quadratic equation. Care should be taken to check solutions, as it is possible that an apparent solution of the resulting quadratic equation makes one or more of the fractions undefined.

Example: Solve for x:

$$\frac{2x}{x+5} + \frac{1}{x-5} = \frac{10}{x^2-25}$$

$2x(x-5) + 1(x+5) = 10$

$2x^2 - 10x + x + 5 = 10$

$2x^2 - 9x - 5 = 0$

$(2x + 1)(x - 5) = 0$

$2x + 1 = 0 \qquad x - 5 = 0$

$2x = -1 \qquad x = 5$

$x = -\frac{1}{2}$

If $x = 5$, the denominators $x - 5$ and $x^2 - 25$ are zero. Therefore $x = 5$ must be rejected. Hence, $x = -\frac{1}{2}$ is the only answer to the given equation.

5. Equations with Radicals

The study of equations with radical expressions will probably be new to the students in Course III. It is important that the students understand that the method of solution — squaring both sides of the equation — may produce an equation whose solutions are not equivalent to that of the original, i.e., $(x = y) \rightarrow (x^2 = y^2)$ but not conversely. By checking the solutions, the students will be able to determine whether or not extraneous solutions have been introduced by the squaring process.

Example:

$\sqrt{9 - 2x} = x - 3$

$9 - 2x = x^2 - 6x + 9$

$0 = x^2 - 4x$

$0 = x(x - 4)$

$x = 0$ or $x = 4$

Check: $\sqrt{9 - 2(0)} \stackrel{?}{} 0 - 3$

$\sqrt{9} \stackrel{?}{} -3$

$3 \neq -3$

$$\sqrt{9 - 2(4)} \;?\; 4 - 3$$
$$\sqrt{1} \;?\; 1$$
$$1 = 1$$

Therefore, the solution set is { 4 } .

Thus the solution of $9 - 2x = (x-3)^2$ does not imply that the values obtained are also solutions of $\sqrt{9 - 2x} = x-3$. Thus, the check becomes very important.

D. REVIEW AND EXTENSION OF SOLUTIONS OF INEQUALITIES

After a review of the techniques used to solve linear inequalities, the students should be introduced to the solution of inequalities involving absolute value or quadratic expressions. The ideas of conjunction, disjunction, and implication can be used to advantage in the presentation of this material.

1. Linear Inequalities

In Course I, the students were introduced to the solution of linear inequalities. The students should be reminded that multiplication or division of an inequality by a negative number changes the sense of the inequality.

If $b < c$ and $a < 0$, then $ab > ac$.

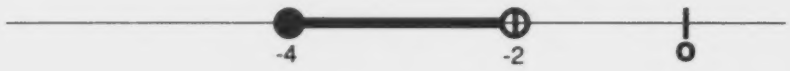
Example: $-3x + 2 < 11$

The solution is $\{ x : x > -3 \}$ and can be represented on the number line as:

A compound inequality such as $8 < -3x + 2 \leq 14$ can be considered a conjunction of two simpler inequalities:

$8 < -3x + 2 \;\wedge\; -3x + 2 \leq 14$

The solution can be represented on the number line as a segment, open at one end and closed at the other end.



2. Absolute Value Inequalities

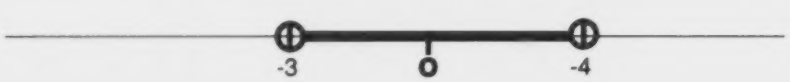
By using the definition of absolute value and the ideas of logic, an inequality containing an absolute value expression can be solved.

Example: $|2x - 1| < 7$

If $2x-1 \geq 0$, then $2x - 1 < 7$. If $2x - 1 < 0$, then $-(2x - 1) < 7$, or $2x - 1 > -7$.

Therefore, $x < 4$ AND $x > -3$.

This solution graphed on the number line is:



3. Quadratic Inequalities

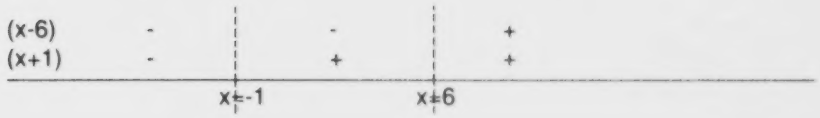
From their work with quadratic equations, the students are familiar with the zero-product property, i.e., if $ab = 0$, then at least one of the factors is equal to 0. In solving quadratic inequalities of the form $ab < 0$, $ab \leq 0$, $ab > 0$ or $ab \geq 0$, the students must

consider that a positive product arises when multiplying two factors with like signs, and that a negative product arises when multiplying two factors with unlike signs.

Example: $x^2 - 5x - 6 < 0$

Solution: By factoring, $(x - 6)(x + 1) < 0$. The two factors must represent numbers with unlike signs. There are two possibilities: either $(x - 6)$ is negative and $(x + 1)$ is positive, or $(x - 6)$ is positive and $(x + 1)$ is negative.

Recording the signs of each factor on a number line will facilitate finding the solution.

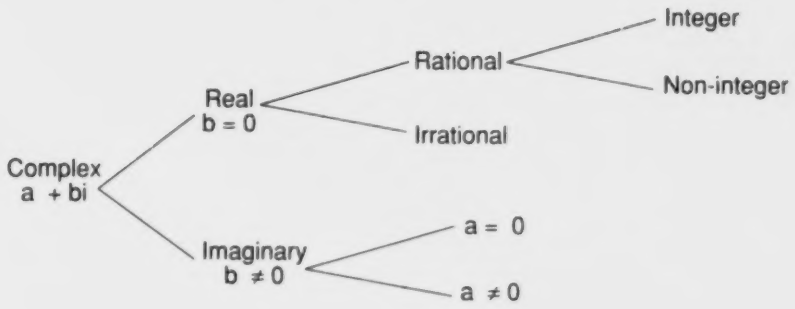


$(x - 6)(x + 1) > 0$ $(x - 6)(x + 1) < 0$ $(x - 6)(x + 1) > 0$

Thus the solution is $-1 < x < 6$.

E. COMPLEX NUMBERS

The fact that many quadratic equations (for example, $x^2 = -1$) have no solution in the set of real numbers motivates the introduction and definition of the complex number system. The students should verify that the complex numbers together with the operations of addition and multiplication form a field. The chart below shows the subsets of the complex numbers.



1. The Imaginary Unit

The imaginary unit, i, is defined as $i = \sqrt{-1}$.

Thus $i^2 = -1$, and the solution to the equation $x^2 = -1$ is $\pm i$. The students should be able to simplify square roots involving a negative radicand.

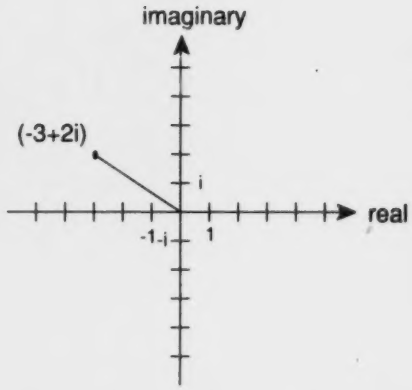
Example: $\sqrt{-128} = 8i\sqrt{2}$

2. Standard Form

Complex numbers are considered in standard form when expressed as a+bi where a and b are real numbers. This description lends itself to a graphic interpretation using

the complex plane with the horizontal axis as the real axis, and the vertical axis as the imaginary axis.

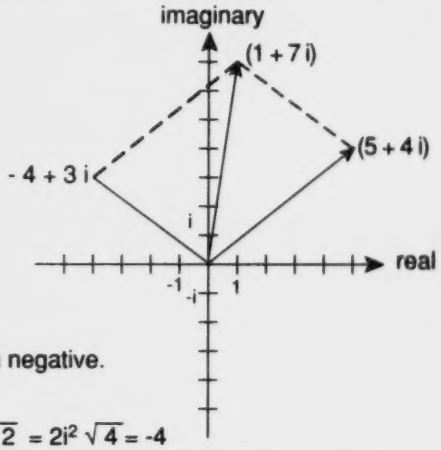
The absolute value of a + bi is defined to be $\sqrt{a^2 + b^2}$. Note that in the graphic representation, $|-3 + 2i|$ is the distance of the point from the origin, $\sqrt{13}$.



3. *Basic Operations*

The basic operations of addition and multiplication and their inverse operations should be defined.

The sum of (a + bi) and (c + di) is (a + c) + (b + d)i. This can be shown graphically as the diagonal of the parallelogram determined by (a + bi) and (c + di). In the diagram, by translating (5 + 4i) so that the point at the origin corresponds to the point (-4 + 3i), the sum of the two complex numbers is shown to be (1 + 7i).



When introducing the product, the following should be noted:

1. $\sqrt{a} \cdot \sqrt{b} \neq \sqrt{a\,b}$ when a and b are both negative.

   Example: $\sqrt{-8} \cdot \sqrt{-2} \neq \sqrt{16}$

   $\sqrt{-8} \cdot \sqrt{-2} = 2i\sqrt{2} \cdot i\sqrt{2} = 2i^2\sqrt{4} = -4$

2. The product of a complex number a + bi and its conjugate a - bi is a real number.

   $(a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2$

This property of the product of a complex number and its conjugate is used in expressing the quotient of two complex numbers in standard form.

Example: Express $\dfrac{3 + 2i}{1 - i}$ in standard form.

Solution:
$$\frac{3 + 2i}{1 - i} \cdot \frac{1 + i}{1 + i} = \frac{3 + 5i + 2i^2}{1 - i^2}$$
$$= \frac{1+5i}{2}$$
$$= \frac{1}{2} + \frac{5i}{2}$$

The cyclic nature of the powers of i should be explored.

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = i^2 \cdot i = -i$

$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

$i^5 = i^4 \cdot i = 1(i) = i$

etc.

Example: Simplify i^{47}

$i^{47} = (i^4)^{11}\, i^3 = 1(-i) = -i$

This cyclic property of the powers of i demonstrates the relation between the powers of i and a rotation of 90°.

4. *Solution of Quadratic Equations*

Quadratic equations... $ax^2 + bx + c = 0$ (a,b,c real numbers and $a \neq 0$) ...should be considered in which the roots are complex numbers. In reviewing the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, it should be noted that if $b^2 - 4ac < 0$, the roots will be imaginary.

The expression $b^2 - 4ac$ is known as the *discriminant* and can be used to describe the nature of the roots of the equation. Roots of a quadratic equation may be classified as real or imaginary, rational or irrational, equal or unequal.

The formulas for the sum ($S = -b/a$) and the product ($P = c/a$) of the roots of a quadratic equation provide a good means for checking the solutions, and can be used to write the equation of a quadratic whose roots are known.

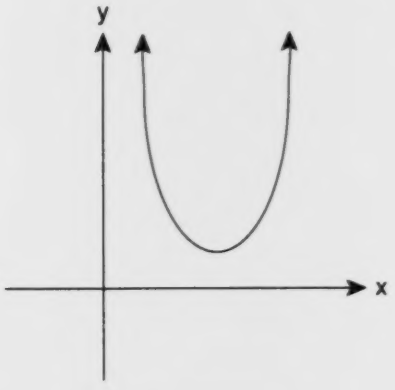
Example: Find a quadratic equation whose roots are $2 - 3i$ and $2 + 3i$.

Solution: The sum of the roots is 4. The product of the roots is 13. Thus, $-b/a = 4$ and $c/a = 13$.

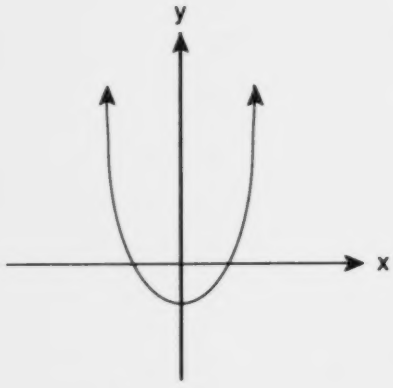
One possible solution can be obtained by letting $a = 1$. Then, $b = -4$ and $c = 13$ satisfy the above.

Therefore, the equation $x^2 - 4x + 13 = 0$ will have the given roots.

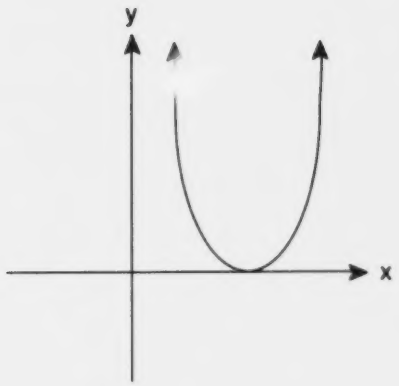
The study of the quadratic should include the relationship between the graph of the quadratic function ($y = ax^2 + bx + c$) and the roots of the quadratic equation ($ax^2 + bx + c = 0$).



$ax^2 + bx + c = 0$ has complex roots.



$ax^2 + bx + c = 0$ has two different real roots.



$ax^2 + bx + c = 0$ has
two equal real roots.

II.
Transformations

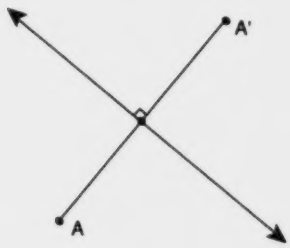
In Course II, the intuitive concepts were reviewed, formalized, and applied to transformations of the coordinate plane. In Course III, transformations will be reviewed again and related to functions and graphing, especially using the relationships among the equations of curves of the same shape.

A. REVIEW

The students should review the basic concepts of line reflections, rotations, translations, and dilations.

1. Line Reflections

A reflection in line ℓ (suggested notation: r_ℓ) is a mapping which takes every point A of the plane into a point A' so that $\overline{AA'}$ is perpendicular to ℓ and the distance from A to ℓ equals the distance from ℓ to A'.



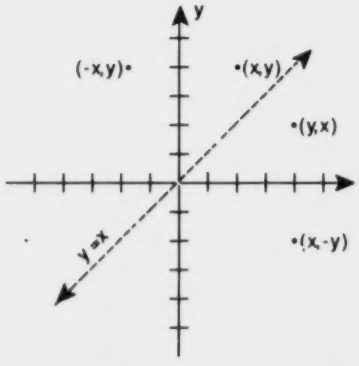
Line reflection preserves distance as well as betweenness, angle measure, and parallelism. Line reflection does not, however, preserve *order* (orientation).

The following relationships between points of the coordinate plane and their images under a line reflection should be developed.

in the x-axis, $(x, y) \rightarrow (x, -y)$

in the y-axis, $(x, y) \rightarrow (-x, y)$

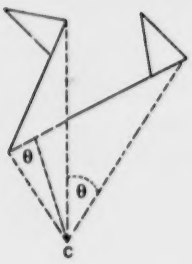
in the line $y = x$, $(x, y) \rightarrow (y, x)$



2. Rotation

A rotation R about center, c, and through an angle of θ (notation: $R_{c,\theta}$ or $Rot_{c,\theta}$) is a mapping which "moves" all points in the plane through an angle of θ with point c fixed at the vertex of the angle. (If θ is positive, the motion is counterclockwise. If θ is negative, the motion is clockwise.)

The center of the rotation is mapped to itself.



It should be verified that this definition results in the following relationships between points of the coordinate plane and their images under a rotation about the origin:

of 90°, $(x, y) \rightarrow (-y, x)$

of 180°, $(x, y) \rightarrow (-x, -y)$

of 270° $(x, y) \rightarrow (y, -x)$

A rotation of 180 degrees about a point P is sometimes called a *half-turn* (equivalent to a reflection in point P).

A rotation of -90° is equivalent to a rotation of 270°.

Examples in the coordinate plane should be limited to rotations through multiples of 90°.

3. Translations

A *translation* is a mapping which assigns to every point in the plane another point in the plane so that every image point is located the same distance away, and in the same direction, from its pre-image, as every other. Or, more simply, a translation "moves" every point in the plane the same distance in the same direction.

It should be verified that this definition results in the following relationship between points of the coordinate plane and their images under a translation of a in the x-direction and b in the y-direction:

$(x,y) \rightarrow (x + a, y + b)$

The translation ($T_{a,b}$) shifts every point by a distance of $\sqrt{a^2 + b^2}$ along a line with a slope of $\frac{b}{a}$ or, if a = 0, an undefined slope.

4. Dilation

A dilation with center at the origin of a coordinate plane and scale factor k (notation: D_k) assigns to every point A another point A' on ray $\overrightarrow{OA}$ so that OA' = k(OA). Point O is mapped onto itself.

It should be verified that this definition results in the following relationship between points of the coordinate plane and their images under a dilation with center at the origin and constant of dilation of scale factor k:

$(x, y) \rightarrow (kx, ky)$

B. COMPOSITION

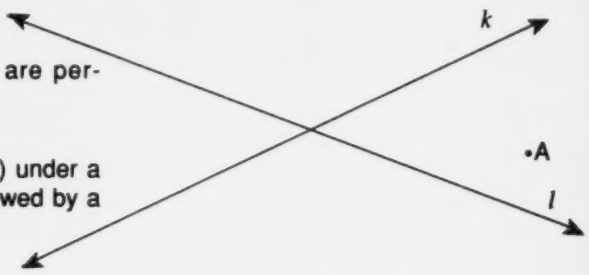
1. General Concepts and Symbolism

The composition of transformations is the result of a second transformation performed on the image of a first.

If r_k is a reflection in line k and r_ℓ is a reflection in line ℓ, then $r_\ell \circ r_k$ is a reflection in line k followed by a reflection in line ℓ.

If $r_k(A) = A'$ and $r_\ell(A') = A''$, then

$r_\ell \circ r_k(A) = r_\ell(A') = A''$.



Notice that the transformations are performed from right to left.

Example:

What is the image of (5, 1) under a reflection in the x-axis followed by a half turn about the origin?

Solution:

$$(5, 1) \xrightarrow{r_{x\text{-axis}}} (5, -1) \xrightarrow{R_{o,180}} (5, -1)$$

or $R_{o,180} \circ r_{x\text{-axis}}(5, 1) = R_{o,180}(5, -1) = (-5, 1)$

Notice that the composition $R_{o,180} \circ r_{x\text{-axis}}$ is the reflection in the y-axis.

The teacher should note that symbolism in transformation geometry is not standard. Some commonly used symbols are:

r_ℓ or R_ℓ for a reflection in line ℓ,

$R_{p,\theta}$ or $Rot_{p,\theta}$ for a rotation of θ about P,

R_{90} or Rot_{90} for a rotation of 90° about the origin.

2. Line Reflections

The composition of two line reflections is a translation if the two lines are parallel.

Example:

The vertices of Δ ABC are A(4, 1), B(1, 2), and C(3, 5). Find the coordinates of the image of Δ ABC under the composition $r_{y\text{-axis}} \circ r_{x=1}$.

Solution:

$$(4, 1) \xrightarrow{r_{x=1}} (-2, 1) \xrightarrow{r_{y\text{-axis}}} (2, 1)$$
$$(1, 2) \longrightarrow (1, 2) \longrightarrow (-1, 2)$$
$$(3, 5) \longrightarrow (-1, 5) \longrightarrow (1, 5)$$

Notice that the solution given above may be summarized as:

$$r_{y\text{-axis}} \circ r_{x=1}(x, y) = (x - 2, y)$$

or

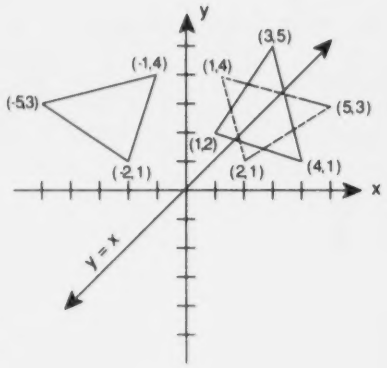
$$r_{y\text{-axis}} \circ r_{x=1} = T_{-2,0}$$

Any translation can be written as the composition of two reflections in parallel lines. The lines of reflection are not unique, but the lines are perpendicular to the direction of the translation, and the distance between the two lines of reflection is one-half the distance between any point and its image under the translation.

The composition of two line reflections will be a rotation if the two lines intersect. The point of intersection is the fixed point of the rotation.

Example:

The vertices of Δ ABC are A(4, 1), B(1, 2), and C(3, 5). Find the coordinates of the image of Δ ABC under the composition $r_{y\text{-axis}} \circ r_{y=x}$.



Solution:

	$r_{y=x}$		$r_{y\text{-axis}}$	
(4, 1)	$\longrightarrow$	(1, 4)	$\longrightarrow$	(-1, 4)
(1, 2)	$\longrightarrow$	(2, 1)	$\longrightarrow$	(-2, 1)
(3, 5)	$\longrightarrow$	(5, 3)	$\longrightarrow$	(-5, 3)

Notice that the solution given above may be summarized as:

$$r_{y\text{-axis}} \circ r_{y=x}(x, y) = (-y, x)$$

or

$$r_{y\text{-axis}} \circ r_{y=x} = R_{O,90}$$

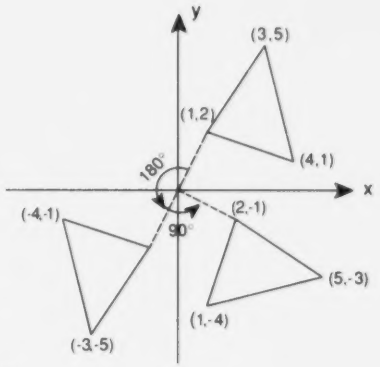
Any rotation can be written as the composition of two reflections over intersecting lines. The lines of reflection are not unique, but the measure of the angle between the two lines of reflection is one-half the measure of the angle of rotation.

3. Rotations

The composition of two rotations about the same center is a rotation. The common center of the two rotations is the center of the composition, and the sum of the measures of the two rotations is the measure of the composition.

Example:

The vertices of Δ ABC are A(4, 1), B(1, 2), and C(3, 5). Find the coordinates of the vertices of the image of Δ ABC under the composition $R_{O,90} \circ R_{O,180}$.



Solution:

	$R_{O,180}$		$R_{O,90}$	
(4, 1)	$\longrightarrow$	(-4, -1)	$\longrightarrow$	(1, -4)
(1, 2)	$\longrightarrow$	(-1, -2)	$\longrightarrow$	(2, -1)
(3, 5)	$\longrightarrow$	(-3, -5)	$\longrightarrow$	(5, -3)

Notice that the above solution may be summarized as:

$$R_{O,90} \circ R_{O,180}(x, y) = (y, -x)$$

or

$$R_{O,90} \circ R_{O,180} = R_{O,270}$$

After the students have studied trigonometric functions, the investigation of rotations through angles whose measures are not multiples of 90° may be pursued as an optional topic.

4. *Translations*

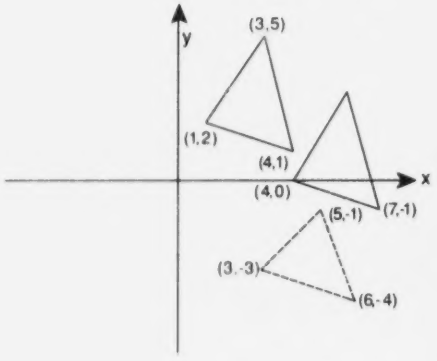
The composition of two translations is also a translation.

Example:

The vertices of Δ ABC are A(4, 1), B(1, 2), and C(3, 5). Find the coordinates of the vertices of the image of Δ ABC under the composition $T_{1,3} \circ T_{2,-5}$.

Solution:

$$\begin{array}{ccccc} & T_{2,-5} & & T_{1,3} & \\ (4, 1) & \longrightarrow & (6, -4) & \longrightarrow & (7, -1) \\ (1, 2) & \longrightarrow & (3, -3) & \longrightarrow & (4, 0) \\ (3, 5) & \longrightarrow & (5, 0) & \longrightarrow & (6, 3) \end{array}$$



Notice that the above solution may be summarized as:

$$T_{1,3} \circ T_{2,-5}(x, y) = (x + 3, y - 2)$$

$$T_{1,3} \circ T_{2,-5} = T_{3,-2}$$

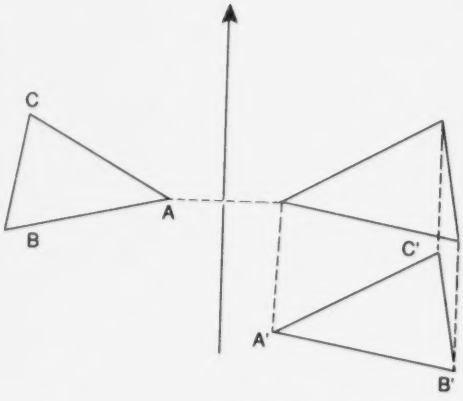
In general:

$$T_{c,d} \circ T_{a,b} = T_{a+c,b+d}$$

5. *Glide Reflections*

The composition of a line reflection and a translation in a direction parallel to the line of reflection is called a glide reflection. Although composition is not always a commutative operation, this composition is commutative.

Since a translation is the composition of two line reflections in parallel lines, a glide reflection can be considered as the composition of three line reflections.



C. ISOMETRIES

1. *Direct*

The students should be encouraged to observe the properties of figures that are preserved under the transformations that they are studying. For all transformations except the dilation, the images under the transformation are congruent to their preimages. Reflections, rotations, translations, and glide reflections are isometries, that is, transformations that preserve distance. Since distance is preserved, angle measure, parallelism, and collinearity are also preserved.

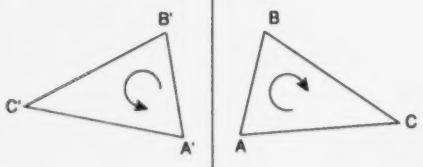
In addition, a direct isometry preserves orientation. That is, if any three noncollinear points are oriented in a clockwise direction, their images under a direct isometry will also be oriented in a clockwise direction. Translations and rotations are direct isometries.

Since distances are not preserved in dilations, they are not isometries. One exception is D_1, which maps every point in the plane onto itself and is therefore the identity; the other is D_{-1}, which is equivalent to a half-turn since it also preserves distance.

2. Opposite

An opposite or nondirect isometry is one that changes the orientation of points from clockwise to counterclockwise or from counterclockwise to clockwise. Reflections and glide reflections are opposite isometries.

Inasmuch as an opposite isometry changes orientation, the composition of two opposite isometries is a direct isometry. For example, the composition of two reflections in parallel lines (opposite isometries) is a translation (a direct isometry).



D. APPLICATION TO GRAPHING

The teacher and the students should view the study of transformations as the development of a useful tool for the investigation of other mathematical topics. The teacher is encouraged to review transformations early in the course and to employ the principles in the study of other topics. Suggestions for doing this will be found in other sections as they occur.

The students should be able to identify graphs that are symmetric with respect to the x-axis, the y-axis, or the origin.

Example:

Show that the graph of $y = x^2$ is symmetric with respect to the y-axis.

Solution: Under a reflection in the y-axis,

$(x, y) \longrightarrow (-x, y)$

$y = x^2 \longrightarrow y = (-x)^2$

$y = x^2 \longrightarrow y = x^2$

Since the image of any point of the parabola under a reflection in the y-axis is a point of the parabola, the graph is symmetric with respect to the y-axis.

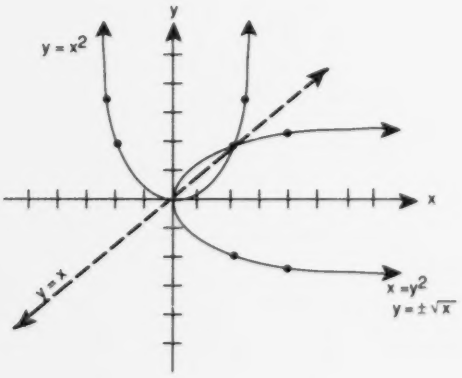
1. Inverse Functions

In the section on functions, reflections can introduce the study of the inverse function. The image of the graph of a function after a reflection in the line $y = x$ may or may not be an inverse.

Example 1:

$y = x^2$

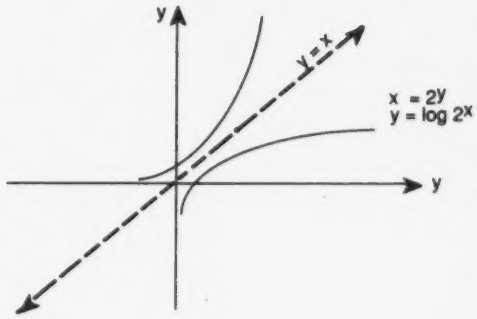
The reflection of the graph of $y = x^2$ in the line $y = x$ is not the graph of a function.



Example 2:

$y = 2^x$

The reflection of the graph of $y = 2^x$ in the line $y = x$ is the graph of a function.



2. *Symmetries*

The students should review the ideas of symmetry from Course I and from Course II. Symmetries with respect to an axis, the origin, or some other point can be used to simplify the graphing of functions.

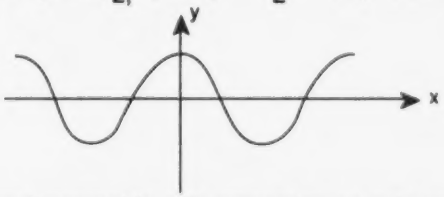
Example:

The graph of $y = \cos x$ has line symmetry under the following transformations:

(1) reflection in the y-axis or the line $x = k\pi$ for integral values of k.

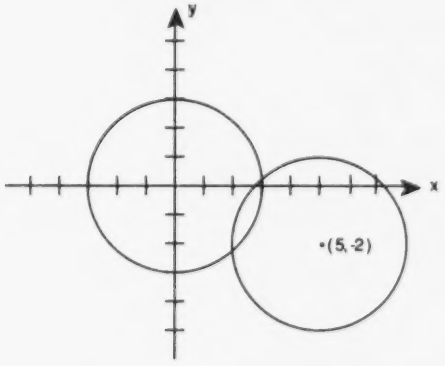
(2) translation $(2\pi, 0)$ or $(2\pi k, 0)$ for integral values of k.

(3) half-turn about the point $(\frac{\pi}{2}, 0)$ or $\left(\frac{(2k+1)\pi}{2}, 0\right)$ for integral values of k.



Some graphs that are not symmetric with respect to an axis can be translated so that the curve has symmetry with respect to the x-axis, the y-axis, or both.

Example:



The translation $T_{5,-2}$ relates the graph of $x^2 + y^2 = 9$ and the graph of $(x - 5)^2 + (y + 2)^2 = 9$.

The symmetries present in the unit circle used to define the trigonometric functions make it convenient to use reflections in the y-axis, the origin, and the x-axis to express the trigonometric function values of any angle in terms of the corresponding function value of an acute angle. (See section V.)

III.
Relations and Functions

The basic concepts of relations and functions provide a central and unifying theme for the study of much of mathematics.

The students are familiar with the concept of relations from their work with algebraic expressions. A relation is implied in an algebraic expression such as 4x - 5, i.e., for each value of x, there is an associated value 4x - 5. Functional notation simplifies the expression of many mathematical ideas. Much of the material in Course III centers on the study of functions: exponential, logarithmic, and trigonometric.

A. RELATIONS

1. Definition

A relation is a set of ordered pairs. The relation may be specified by listing the ordered pairs (if the set of elements is finite), or may be specified by a rule making it possible to determine the ordered pairs belonging to the relation.

Example 1:

Student	*Score*
Mary	87
Joe	94
Peter	82
Alice	94

Example 2: $R = \{ (x,y) \mid y = \sqrt{x - 1} \}$

Example 3: $y = 4x - 5$

2. Domain and Range

The *domain* of the relation is the set of all first components of the ordered pairs, and the *range* of the relation is the set of all second components of the ordered pairs.

In Example 1 above, the domain is { Mary, Joe, Peter, Alice } and the range is { 87, 82, 94 }. In Example 2, the domain is the set of real numbers greater than or equal to 1, i.e., $\{ x \mid x \geq 1 \}$, and the range is the set of nonnegative reals, i.e., $\{ y \mid y \geq 0 \}$. In Example 3, both the domain and the range are the set of real numbers.

If the domain of a relation is not specified, it is assumed to be the largest subset of the reals for which the range consists only of real numbers.

B. FUNCTIONS

1. Definition

A function is a relation in which each element of the domain corresponds to exactly one element of the range. In other words, a function is a relation in which no two ordered pairs have the same first element.

Example 1:

$y = x^3$ is a function, since for each first element, x, there is exactly one second element, y.

Example 2:

The relation $|y| = x$ does not define y as a function of x, since for each first element, x, there is more than one second element, y.

It is sometimes useful to consider a function as a mapping of one variable to another. The expression $x \rightarrow x^2 - 3$ indicates that each first element, x, is mapped to a second element, $x^2 - 3$.

A common notation for a function is to express the rule for the function in terms of f(x). For example, the linear function $y = 1/2\ x + 4$ can be written as $f(x) = 1/2\ x + 4$. This notation is particularly useful to express ordered pairs associated with the function. If $f(x) = 1/2\ x + 4$, then $f(0) = 4$, $f(2) = 5$, $f(4) = 6$ and $f(a) = 1/2\ a + 4$. The students should become familiar with this notation and use it with a variety of functions throughout the remainder of the course.

The composition of functions of f(g(x)) or $[f \circ g]\ (x)$ should be defined and the importance of the order noted.

Example:

$f(x) = x^2$ and $g(x) = x + 1$

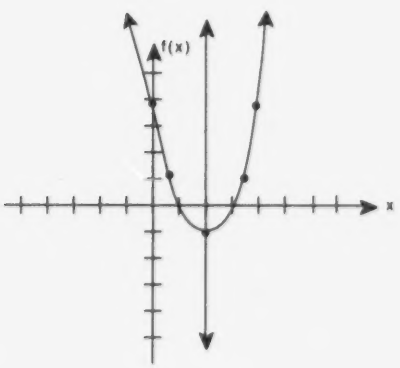
$f(g(x)) = (x + 1)^2$ and $g(f(x)) = x^2 + 1$

2. Determining if a Relation is a Function

The use of the vertical line test, i.e., passing a vertical line across the graph of a relation, provides a quick visual check to determine if a relation is a function. These two examples show the use of this test:

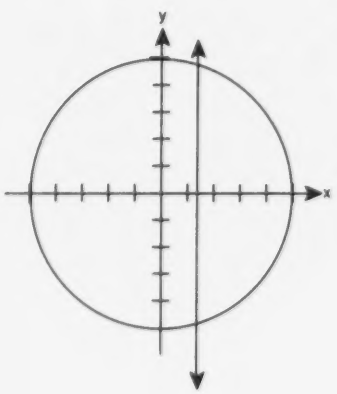
Example 1:

$f(x) = x^2 - 4x + 3$



Example 2:

$y = \pm \sqrt{25 - x^2}$



In Example 1, for each x in the domain, a vertical line will intersect the graph of the relation in one and only one point, whereas in Example 2, for many values of x in the domain the line will intersect the relation in two points. Thus $f(x) = x^2 - 4x + 3$ is a function, but $y = \pm\sqrt{25 - x^2}$ is not a function.

The students should recognize $ax^2 + by^2 = c$ as the equation of a circle, an ellipse, or a hyperbola as determined by the values of a, b, and c. In general, these equations name relations that are not functions. The equation $xy = c$ names a special hyperbola which is a relation that is also a function.

3. *Inverse Variation and the Hyperbola*

If two quantities, x and y, vary inversely, then their product is a constant. Examples of inverse variation can be drawn from science, business, and common experience.

Example 1:

At a constant temperature, the pressure of a gas varies inversely as its volume.

Example 2:

If each person works at the same rate, the number of persons working varies inversely as the time needed to complete a job.

Direct variation can be reviewed and compared with inverse variation. In a given situation, depending on the variable that is held constant, the related variables may vary directly or inversely. In the formula $D = rt$, if D is held constant, then r and t vary inversely; if t is held constant, then D and r vary directly.

Example 1:

When on vacation, the Schwartz family always drives 300 miles a day. The number of hours that they drive (t) varies inversely as the average rate at which they drive (r). Therefore $rt = 300$.

Examples 2:

The Perez family always travels for 6 hours a day. The distance (D) that they travel in a day varies directly as the rate at which they drive (r). $D/r = 6$.

A similar situation exists using the formula $T = nc$.

Example 3:

The number of candy bars (n) that may be purchased with $5.00 varies inversely as the cost of one bar (c). $nc = 5$

Example 4:

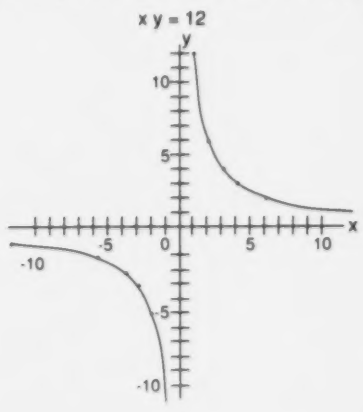
The total cost (T) of 10 bars of candy varies directly as the cost of one bar (c). $T/c = 10$

If c is the constant of variation, then the definition of inverse variation produces the function $xy = c$, or $y = c/x$. The graph of this function is a hyperbola.

Example: Sketch the graph of $xy = 12$.

x	y
-12	- 1
- 6	- 2
- 4	- 3
- 3	- 4
- 2	- 6
- 1	-12

x	y
12	1
6	2
4	3
3	4
2	6
1	12



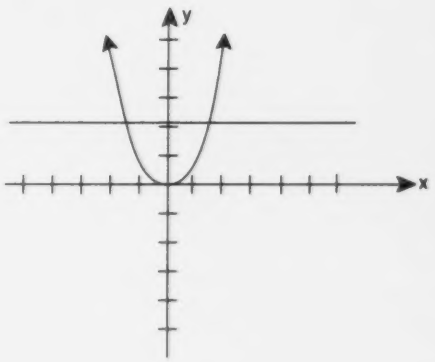
Notice that as the value of one variable increases, the value of the other variable decreases. The students should understand the meaning of inverse variation, and be able to graph this function and recognize the resulting curve as a hyperbola.

4. One-to-One Functions

A function is classified as one-to-one if and only if each second element corresponds to exactly one first element. This concept is important in determining whether or not a function has an inverse.

Example:

$f(x) = x^2$ is *not* one-to-one, since, for example, when $f(x) = 4$, x can be either +2 or -2.



It is sometimes useful to use a horizonal line test to determine if a function is one-to-one, i.e, if a horizontal line intersects the graph of the function in more than one point, then the function is *not* one-to-one.

C. INVERSE FUNCTIONS

1. Definition and Characteristics

The set of ordered pairs obtained by interchanging the first and second elements of each pair belonging to a one-to-one function is called the inverse function, denoted as f^{-1}.

Example:

If $y = 2x - 1$, then the inverse function formed by interchanging x and y is $x = 2y - 1$. Solving for y, $y = \frac{x + 1}{2}$.

Thus if $f(x) = 2x - 1$, then $f^{-1}(x) = \frac{x + 1}{2}$.

The students should note that the figure consisting of the graphs of a function and its inverse function are always symmetric with respect to the line $y = x$.

The result of interchanging the first and second elements of each ordered pair belonging to a function which is not one-to-one is *not* a function and therefore is *not* an inverse.

Example:

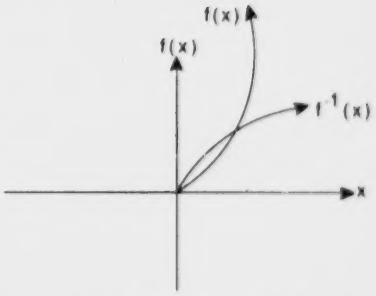
Upon interchanging x and y, $y = x^2$ becomes $x = y^2$, or solving for y, $y = \pm \sqrt{x}$.

The graph of $y = \pm \sqrt{x}$ clearly shows that it is not a function.

A common method of producing an inverse function when the original function is *not* one-to-one is to restrict the domain of the original function to make it one-to-one.

Example:

If the domain of $f(x) = x^2$ is restricted to $x \geq 0$, then $f^{-1}(x) = +\sqrt{x}$.



The students have worked with inverses in other settings and have seen that using an inverse "will undo what has been done." In arithmetic, -3 is the inverse of + 3 under the operation of addition, and +3 + -3 = 0, the additive identity. In a similar manner, 1/3 is the multiplicative inverse of 3, and (3) (1/3) = 1, the multiplicative identity. In much the same way, a function and its inverse "undo each other".

If $f(x) = \frac{x+1}{2}$ then $f^{-1}(x) = 2x - 1$

In the above example, $f(5) = 3$ and $f^{-1}(3) = 5$. More generally,

$$f(f^{-1}(x)) = \frac{(2x-1)+1}{2} = x$$

and $$f^{-1}(f(x)) = \frac{2(x+1)}{2} - 1 = x.$$

For any function and its inverse, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$. The function $f(x) = x$ is known as the identity function.

D. THE EXPONENTIAL FUNCTION

The exponential function has great significance in much of mathematics and science. To prepare for the study of this function, the students need a thorough review and extension of their previous work with exponents.

1. Rational Numbers as Exponents

The students should be familiar with the use of positive, negative, and zero exponents and with the laws used in working with expressions containing exponents. These should be reviewed at this time. The meaning and use of any rational number as an exponent should be considered. If a is a real number, and m and n are positive integers, then

$a^0 = 1, a \neq 0$

$a^1 = a$

$a^m = \underbrace{a \cdot a \cdot a \cdot a \cdot a \ldots a}_{m \text{ factors}}$

$a^{-1} = \frac{1}{a}, a \neq 0$

$a^{-m} = \frac{1}{a^m}, a \neq 0$

$a^m \cdot a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}, a \neq 0$

$(a^m)^n = a^{mn}$

$$a^{\frac{1}{n}} = \sqrt[n]{a}, n \neq 0$$

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m, a > 0, n \neq 0$$

It should be noted that $a^{\frac{1}{n}}$ does not necessarily name a real number if $a < 0$. For example, $(-4)^{1/2} = 2i$.

In the development of the use of exponents, the students should review scientific notation and its use in expressing very large or very small numbers.

A variety of problems should be given to the students to develop their skills in working with expressions involving exponents.

Example 1: Evaluate $27^{2/3}$.

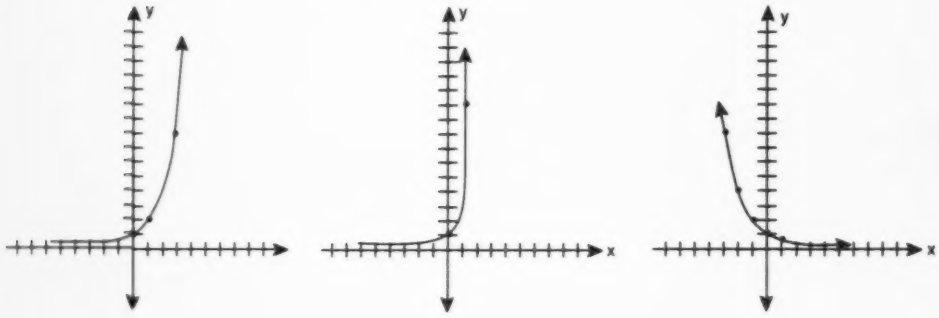
Example 2: Find f(4) if $f(x) = 2x^{1/2} - 5$.

Example 3: Express in scientific notation, 0.0462.

Example 4: Find x if $2^{x+1} = 16$.

2. The Function, $y = a^x$

An exponential function is a function of the form $y = a^x$ or $f(x) = a^x$ where $a > 0$, $a \neq 1$. The students should sketch graphs of several exponential functions and note the nature of these graphs.



$y = 2^x$ $y = 10^x$ $y = \left(\frac{1}{2}\right)^x$

Characteristics to be emphasized include:

(1) The domain of an exponential function is the set of real numbers.

(2) The range of an exponential function is the set of positive numbers.

(3) The graph of any exponential function will contain the point (0,1).

(4) The exponential function is one-to-one.

(5) If $a > 1$, the graph rises, but if $0 < a < 1$, the graph falls.

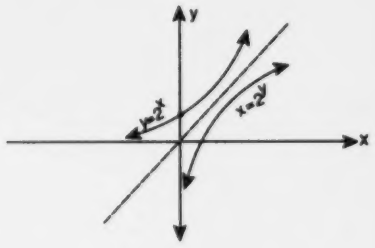
(6) The graphs of $y = a^x$ and $y = a^{-x}$, $a > 0$, and $a \neq 1$, are reflections of each other in the y-axis.

E. THE LOGARITHMIC FUNCTION

1. $y = \log_a x$ as the inverse of $y = a^x$

The logarithmic function should be presented as the inverse of the exponential function.

If $f(x) = a^x$, then $f^{-1}(x) = \log_a x$. For example, if the elements of the ordered pairs belonging to the graph of the exponential function $y = 2^x$ are interchanged, a new graph is obtained such that $x = 2^y$.



The students should become familiar with the common notation used to express y explicitly; that is, $y = \log_a x$. They should graph several logarithmic functions and observe common characteristics.

(1) Since the exponential function is one-to-one, its inverse, the logarithmic function, exists.

(2) The domain of the logarithmic function is the set of positive real numbers.

(3) The range of the logarithmic function is the set of all real numbers.

(4) The graph of any logarithmic function will contain the point (1,0).

(5) The graphs of $y = a^x$ and $x = a^y$, $a > 0$, and $a \neq 1$ are reflections of each other in the line $y = x$.

2. Working with Logarithms

The students need to become familiar with expressions such as $\log_2 8$ and be able to solve equations using logarithmic expressions. They should be able to rewrite the equality $\log_b a = c$ as $a = b^c$.

Example 1: Find $\log_2 64$.

Example 2: Solve for x: $\log_x 81 = 4$.

Example 3: Find x: $\log_5 x = 2$.

3. Common Logarithms

Logarithms in base 10, known as common logarithms, have historical significance as computational aids and as the basis of the slide rule. Although calculators and computers have replaced the need for logarithmic computation, the students should have some experience in this area to develop an understanding of the principles involved in working with logarithms.

Logarithms in base 10 consist of two parts, the characteristic and the mantissa. The students may use scientific notation to find the characteristic and then read the mantissa from a table.

4. Laws Used in Computation with Logarithms

Since logarithms represent exponents, rules for performing calculations involving multiplication, division, raising to powers, and finding roots may be derived from the laws of exponents. The students should understand and be able to use the following laws:

$\log (AB) = \log A + \log B$

$\log (A/B) = \log A - \log B$

$\log (A^n) = n \log A$

$\log \sqrt[n]{A} = \frac{1}{n} \log A$

Example 1:

Using logarithms, find to the nearest 100th, $\sqrt[3]{0.862}$.

Example 2:

Using logarithms, find to the nearest 10th, $\sqrt{\frac{48.6}{\pi}}$.

The complexity of the calculations should be limited to that necessary for the development of the students' ability to apply the laws and to understand the important properties of logarithms. An emphasis should be placed on the functional and graphical aspects of these functions rather than the computational use of logarithms. (Such emphasis is valuable for future studies in mathematics and science.)

5. Applications

In addition to using logarithms for computations involving multiplication, division, powers, and roots, the students should develop an understanding of the use of logarithms in solving certain exponential equations.

Equations such as $4^x = 8^{x-1}$ can be solved without the use of logarithms, since both 4 and 8 can be written as powers of the same base number. Thus, $2^{2x} = 2^{3x-3}$ and $x = 3$.

However, in an equation such as $5^x = 17$, this cannot be done by inspection. Using a logarithmic system such as base 10 will allow for a solution of such an equation. In effect, the students will write both 5 and 17 as powers of 10:

$5^x = 17$

Using common logs, $5 = 10^{0.6990}$ and $17 = 10^{1.2304}$ or $10^{0.6990x} = 10^{1.2304}$

Thus, $0.6990\,x = 1.2304$

and $x = 1.76$

Or, equivalently: $5^x = 17$

$x \log 5 = \log 17$

$x = \frac{\log 17}{\log 5}$

IV.
Circles

This section reviews and extends the work with formal geometric proof begun in Course II. Since many of the theorems suggested for inclusion lead to numerical relationships among the measures of arcs, angles, and segments of chords, tangents, and secants, the problems in this section provide a rich blend of algebra and geometry.

A. ARCS, ANGLES, AND CHORDS

1. Definitions

In Course II, the students learned that in a plane the locus of points equidistant from a fixed point is a circle. This locus concept is basic to the definition of a circle. Definitions of line segment, angle, angle measure, and congruence from Courses I and II should be reviewed, and used to define chord, diameter, central angle, major arc, minor arc, and semicircle.

A postulate or definition of the degree measure of an arc should be presented as the basis of the theorems that relate arc and angle measure. A common approach is to define the degree measure of a minor arc as the measure of the central angle that intercepts that arc.

2. Theorems Related to Congruence

The necessary definitions, postulates, and simple theorems should be presented to enable the students to prove the following theorems:

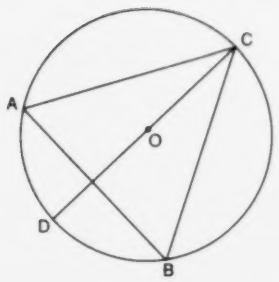
1. In a circle or in congruent circles, if two central angles are congruent then the arcs that they intercept are congruent and conversely.
2. In a circle or in congruent circles, if two central angles are congruent then the chords that they subtend are congruent and conversely.
3. In a circle or in congruent circles, if two arcs are congruent then the corresponding chords are congruent and conversely.
4. If two chords are equidistant from the center of a circle, then they are congruent and conversely.

The students should be able to prove simple theorems and original exercises.

Example 1:

Prove that if a diameter of a circle is perpendicular to a chord it bisects that chord.

Example 2:



Given: ΔABC in circle 0,

$\overline{CD}$ is a diameter

and $\overline{AC} \cong \overline{BC}$

Prove: $\overline{CD} \perp \overline{AB}$

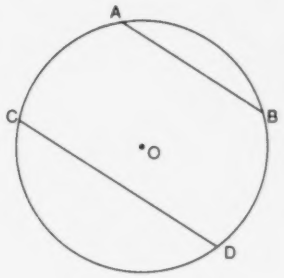
Example 3: In a circle, chord $\overline{AB}$ is 9 cm from the center of the circle. If the length of $\overline{AB}$ is 24 cm, find the length of the radius of the circle.

3. Theorems Related to Angle Measure

The following theorems should be proven and used in the proofs of other theorems as well as in the solution of numerical problems.

1. The degree measure of an inscribed angle is equal to one-half the degree measure of its intercepted arc.

Example:



Given: Circle 0

$\overset{\frown}{AC} \cong \overset{\frown}{BD}$

Prove: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

2. The degree measure of an angle formed by two chords intersecting within a circle is equal to one-half the sum of the degree measures of the arcs intercepted by the angle and by its vertical angle.

B. CHORDS, SECANTS, AND TANGENTS

1. Definitions

A line in the plane of a circle can intersect the circle in 2, 1, or 0 points. If a line intersects the circle in 2 points, it is a secant of the circle. The segment of the secant that is within the circle is a chord of the circle. If a line intersects the circle in 1 point, it is tangent to the circle, and the point is the point of tangency. The perpendicular relationship between a tangent line and a radius drawn to the point of tangency can be established in a variety of ways that depend upon the definitions and postulates presented.

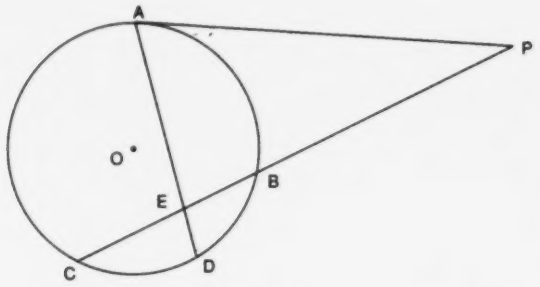
2. Theorems Related to Angle Measure

The following theorem should be proved and used to prove other theorems.

1. The degree measure of an angle formed by a tangent and a chord is equal to one-half the degree measure of its intercepted arc.

2. The degree measure of an angle formed by two secants, two tangents, or a secant and a tangent intersecting outside the circle, is equal to one-half the difference of the degree measures of the intercepted arcs.

Example:



In the diagram, $\overleftrightarrow{PA}$ is tangent to circle 0 at A; $\overset{\frown}{mBD}:\overset{\frown}{mDC}:\overset{\frown}{mCA}:\overset{\frown}{mAB} = 1:2:4:3$.

Find m $\angle$ AEB and m $\angle$ P.

3. *Theorems Related to Lengths of Segments*

The following theorems should be proved and used to prove other theorems:

1. If two chords intersect within a circle, the product of the measures of the segments of one chord is equal to the product of the measures of the segments of the other.
2. Two tangent segments drawn to a circle from an external point are congruent.
3. If a tangent and a secant are drawn to a circle from an external point, the measure of the tangent segment is the mean proportional (geometric mean) between the measures of the secant and its external segment.
4. If two secants are drawn to a circle from an external point, the product of the measures of one secant and its external segment is equal to the product of the measures of the other secant and its external segment.

Example 1:

Prove that the bisector of the angle formed by two tangents to a circle passes through the center of the circle.

Example 2:

Tangent $\overleftrightarrow{AB}$ and secant $\overleftrightarrow{ACD}$ are drawn to circle 0 from external point A. If AB = 6 and CD = 5, find AD.

Solution: Let $x = AC$

$x + 5 = AD$

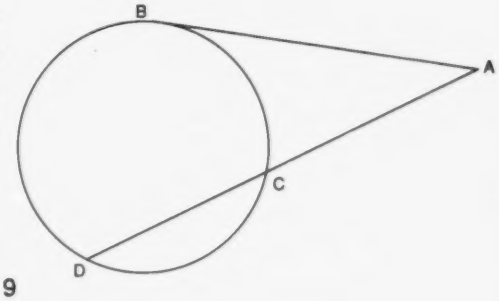
$x(x + 5) = 36$

$x^2 + 5x = 36$

$x^2 + 5x - 36 = 0$

$(x + 9)(x - 4) = 0$

$x = -9$ or $x = 4$



Since AC is positive, AC = 4 and AD = 9

Example 3:

$\overline{PA}$ and $\overline{PB}$ are tangent to circle 0 at A and B respectively. If PA = 10m and m $\angle$ APB is 34, find the length of chord $\overline{AB}$ to the nearest meter. (A problem such as this can be presented after the students have studied the Law of Cosines.)

V.
Trigonometric Functions and Their Inverses

In Course II, the students studied right triangle trigonometry, and expressed trigonometric function values as the ratio of the measures of the sides of right triangles. The students should be aware that these ratios apply only to the special case of a right triangle. In Course III, they will learn the general definitions of the trigonometric functions.

A. DEFINITIONS

1. Angles as Rotations

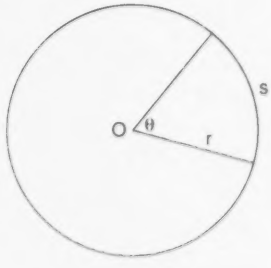
In their study of geometry, the students used a static concept of angles in convex polygonal figures in which the measure of an angle is limited to positive values less than or equal to 180 degrees. In defining the trigonometric functions, the dynamic interpretation of angle from transformation (rotation) is used. Therefore it is possible for any real number to be the measure of an angle.

2. Unit Circle

A unit circle or circle of radius one is a convenient vehicle for defining the trigonometric functions. In the Cartesian coordinate system, a circle with center at the origin and radius one is designated by $\{ (x, y) \mid x^2 + y^2 = 1 \}$.

3. Radian Measure

Although degree measure is convenient in relating angle or arc size to that of a complete rotation, it is often more important to use a measure which relates angle and arc measure to linear measure. If a central angle of a circle intercepts an arc equal in length to the radius of the circle, then the measure of the angle is one radian. In particular, in the unit circle, a central angle of 1 radian intercepts an arc of 1 unit, and a central angle of n radians intercepts an arc of n units. Using dilations, we can show that in a circle of radius r the length of the arc intercepted by an angle of θ radians is θr. Therefore, in circle O, if θ represents the radian measure of a central angle, r represents the radius of the circle, and s represents the length of the arc that subtends the central angle:



$$s = \theta r$$

$$\text{or} \quad \theta = \frac{s}{r}$$

Example 1:

Find the length of the arc that subtends an angle of 2 radians in a circle of radius 5 cm.

Example 2:

Find the length of the arc that subtends an angle of 2 degrees in a circle of radius 5 cm.

The students should be able to express angle measure in terms of either degrees or radians, and should be able to convert from one measure to the other.

Example 1:

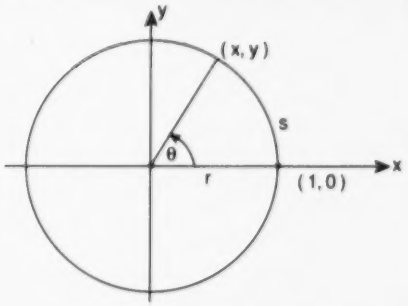
Express in degrees the measure of an angle of $\frac{5\pi}{12}$ radians.

Example 2:

Express in radians the measure of an angle of 100°.

4. Sine and Cosine

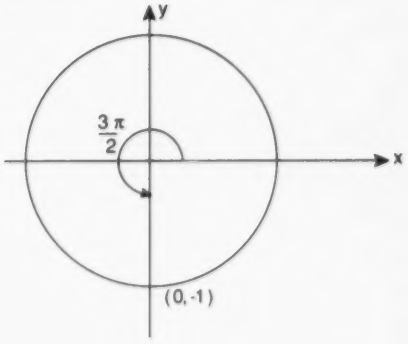
If a ray coincident with the nonnegative x-axis is rotated through θ radians, a point on that ray one unit from the origin traces out a path on the unit circle of length θ. The ray coincident with the nonnegative x-axis and the final position of this ray determine both an arc of the unit circle and a central angle. The ray coincident with the axis is called the initial side of the angle, and the final position of the ray is called the terminal side. Such an angle is said to be in standard position. The point (1, 0) represents the zero position for measuring arc length along the unit circle. Motion in the counterclockwise direction is considered positive, and motion in the clockwise direction is considered negative. For each real number θ which represents the measure of a central angle or the length of path on the unit circle, there is a point (x, y) such that $x^2 + y^2 = 1$.



Sin θ = y means that sin θ is the directed vertical distance from the x-axis to the endpoint of the arc of length θ on the unit circle; that is, the y- coordinate of the point of intersection of the terminal ray and the unit circle.

Cos θ = x means that cos θ is the directed horizontal distance from the y-axis to the endpoint of the arc of length θ on the unit circle; that is, the x- coordinate of the point of intersection of the terminal ray and the unit circle.

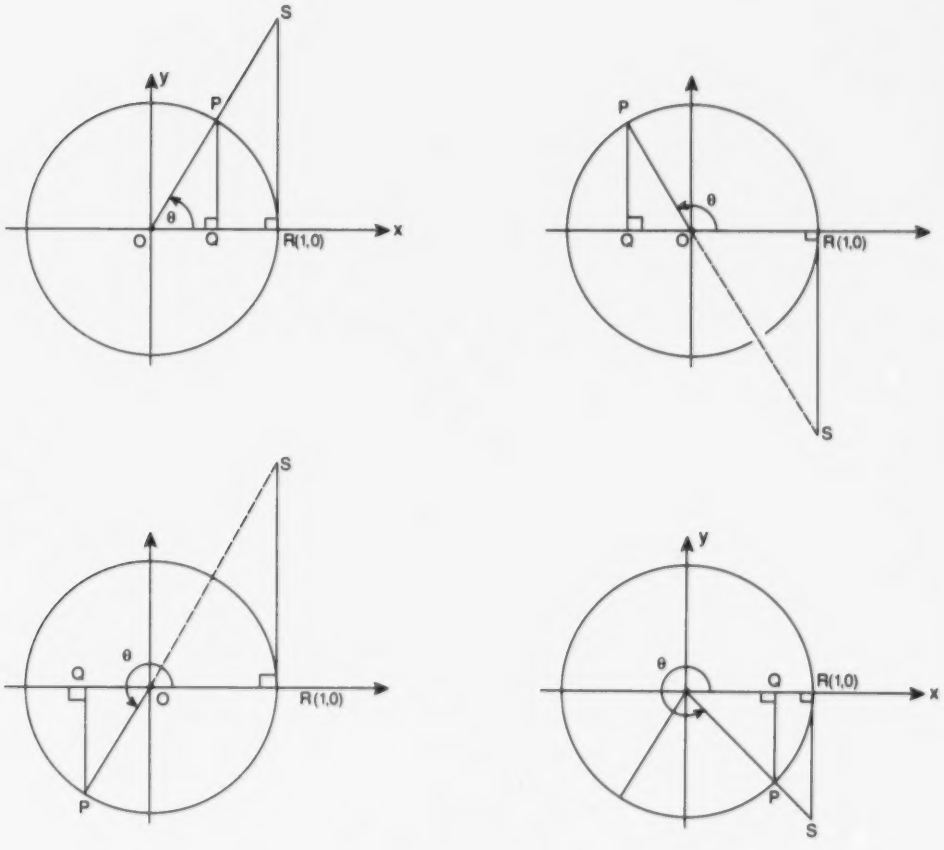
If the terminal ray of an angle coincides with either axis, the angle is called a quadrantal angle. The coordinates of the intersections of the axes with the unit circle give the sines and cosines of angles of 0, $\frac{\pi}{2}$, $\frac{3\pi}{2}$, and 2π radians. The sketch shows that $\sin \frac{3\pi}{2}$ or sin 270° = -1, and that $\cos \frac{3\pi}{2}$ or cos 270° = 0.



5. Tangent

Using an angle of measure θ in standard position in the unit circle whose terminal side intersects the unit circle at (x, y), we can define tan θ as $\frac{y}{x}$. Using similar triangles, we can show that tan θ is the directed distance along the tangent to the unit circle at (1, 0) from the x-axis to the point where the tangent line intersects the terminal side of the angle of measure θ.

The following representation of the function values for the sine, cosine, and tangent functions as directed distances may be helpful to the students.



sin θ = PQ
cos θ = OQ
tan θ = SR

6. Coordinates of a Point

Let P be a point on the unit circle and on the terminal side of an angle of measure θ. Using the definitions of sin θ and cos θ, we can express the coordinates of P as (cos θ, sin θ). If the point does not lie on the unit circle but is a distance *a* from the origin, dilations can be used to show that its coordinates are (a cos θ, a sin θ).

7. Reciprocal Trigonometric Functions

Three additional functions are defined using the reciprocals of the function values of sine, cosine, and tangent.

Cosecant function: $\csc \theta = \frac{1}{\sin \theta}$, $\sin \theta \neq 0$

Secant function: $\sec \theta = \frac{1}{\cos \theta}$, $\cos \theta \neq 0$

Cotangent function: $\cot \theta = \frac{1}{\tan \theta}$, $\tan \theta \neq 0$

B. BASIC IDENTITIES

The definitions of the last three trigonometric functions are called the reciprocal identities:

$$\csc \theta = \frac{1}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta} \qquad \cot \theta = \frac{1}{\tan \theta}$$

From the definition of sine, cosine, and tangent and from these identities, we obtain the quotient identities:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \qquad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

Inasmuch as the sine and cosine functions have been defined as $\cos \theta = x$ and $\sin \theta = y$ on the circle whose equation is $x^2 + y^2 = 1$, it follows that:

$$\cos^2 \theta + \sin^2 \theta = 1$$

By dividing both sides of this identity by $\cos^2 \theta$ or by $\sin^2 \theta$, we obtain:

$$1 + \tan^2 \theta = \sec^2 \theta \text{ if } \cos \theta \neq 0$$

$$\cot^2 \theta + 1 = \csc^2 \theta \text{ if } \sin \theta \neq 0$$

These three are called the Pythagorean Identities.

These eight identities are often called the basic identities of trigonometry, and are used to prove other identities. They can also be used to express any function value in terms of any other function value for the same angle.

Example: Express $\tan \theta$ in terms of $\sin \theta$.

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sin \theta}{\pm \sqrt{1 - \sin^2 \theta}}$$

C. FUNCTION VALUES

1. Special Angles

Angles of 30°, 45°, 60°, and their multiples occur frequently in applications. The students should review the relationships in the 30°–60°–90° and the 45°–45°–90° triangles, and use these relationships to obtain the exact function values for angles of these special measures.

2. Use of Tables

The students should be able to use tables that list the function values associated with angles whose measures are given for every 10 minutes from 0° to 90°.

3. Reference Angles

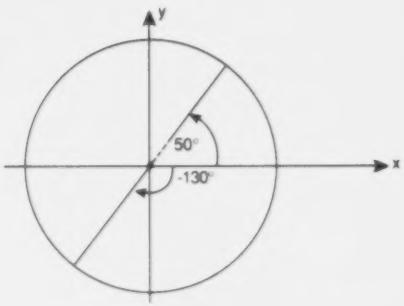
If θ is an angle in a quadrant other than the first, the reference angle for θ is the first quadrant angle whose trigonometric function values have the same absolute value as θ. The relationship between a function value of any angle and its reference angle can be derived using reflections. For a second quadrant angle, a reflection in the y-axis is used. For a third quadrant angle, a reflection in the origin is used. For a fourth quadrant angle, a reflection in the x-axis is used.

If the measures of angles differ by a multiple of 2π radians or 360°, then they have the same terminal side and are called coterminal angles.

The students should be able to use reference angles and coterminal angles to find the function values for angles of any measure.

Example:

Express sin (-130) as a function of a positive acute angle.



D. GRAPHS

1. Graphs of Basic Functions

The students should be able to graph the basic trigonometric functions $y = \sin x$, $y = \cos x$, and $y = \tan x$. In the sketching of the graphs of these functions, the measure of the angle x should be expressed in radians, and the scale on the x-axis made comparable to that used on the y-axis.

2. Amplitude and Period

The significance of a and b in the expressions $y = a \sin bx$ and $y = a \cos bx$ should be understood and used in graphing these functions. The meaning of amplitude and period should be presented. Investigation should be made of the changes brought about by reflections and translations and the symmetries that exist with respect to reflections and translations.

The trigonometric functions are the first examples of periodic functions that the students have encountered. They should be able to recognize a periodic function as one which is symmetric under the translation $T_{p,0}$ when p is the period of the function or a multiple of the period of the function.

E. RELATED RELATIONS AND FUNCTIONS

1. Reflections in the Line y = x

When the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$ are reflected in the line $y = x$, the results are relations which are not functions. These relations are usually expressed as $y = \arcsin x$, $y = \arccos x$, and $y = \arctan x$. The students should understand and make use of this notation. (Alternate notation used is $y = \sin^{-1}x$, $y = \cos^{-1}x$, $y = \tan^{-1}x$.)

Example:

Express the roots of $2 \sin x + 1 = 0$ in terms of arc sin.

Solution: $x = \text{arc sin}\left(-\frac{1}{2}\right)$.

2. *Inverse Functions*

By restricting the domain of $y = \sin x$ to $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$, of $y = \cos x$ to $0 \leq x \leq \pi$, and of $y = \tan x$ to $\frac{-\pi}{2} < x < \frac{\pi}{2}$, we find that the results of reflecting these subsets of the trigonometric functions in the line $y = x$ are functions. The students should be familiar with these domains and the resulting ranges. The inverse trigonometric functions are usually written as $y = \text{Arc sin } x$, $y = \text{Arc cos } x$, and $y = \text{Arc tan } x$.

Example 1:

If $x = \text{Arc cos } \frac{-\sqrt{3}}{2}$, find x.

Solution:

$x = 150^\circ$ or $x = \frac{5\pi}{6}$.

Example 2:

Find cos (Arc tan(-1)).

Solution: $\frac{\sqrt{2}}{2}$.

VI.
Trigonometric Identities and Equations

With their background in basic trigonometry, the students should now be ready to prove trigonometric identities and to solve equations whose variables are trigonometric functions. The teacher may wish to review the basic concepts of transformation geometry; they offer an efficient approach to many of the trigonometric ideas of this unit.

A. IDENTITIES

The demonstration that a given open sentence is an identity, true for all permissible replacements of the variable, is an important part of the students' experience with trigonometry. In this work, it is essential to emphasize the basic difference between equations in general and identities. An equation is simply a statement of equality of two quantities: it may be true for some, all, or no replacements of the variable. The equations most familiar to the students are of the first kind, for example, $x^2 = 9$, which is true for only two real numbers. An example of an equation true for *no* replacements of the variable is $y + 1 = y + 3$. An equation true for *all* replacements of the variable is called an identity: for example, $(x-2)(x+2) = x^2 - 4$ is a statement which is true for all real numbers x.

The students should understand that when they "prove a trig identity" they are not just finding a few values which make the equation true, but are showing that the equation is true for all permissible values. An identity can be said to be an equation which, when universally quantified, is true. Some teachers may wish to use the universal quantification symbol, $\forall_x$, "for all x," in their statements of identities to be proved.

1. Review and Extension of Basic Relationships

Before proceeding with this section, the teacher should review the definitions of sine, cosine, and tangent, the reciprocal identities, and the Pythagorean Identities. The teacher may also wish to go over the students' recently acquired skills with algebraic fractions. The notions of "identity" and "proof" should also be reviewed.

The following two problems are typical of those which might be solved as preparation for more advanced work. The solution of the second problem involves application of the students' recently studied work with complex fractions.

Examples: If possible, write each of the following in terms of a single trigonometric function, or a constant.

Example 1. $\sec x \sin x \cot x$

Solution: 1

Example 2. $\dfrac{\cot B}{\csc B}$

Solution: $\cos B$

2. Proving Trigonometric Identities

The quotient identities, the reciprocal identities, and the Pythagorean Identities can be used to establish that other equations are identities. There are a variety of techniques for approaching such problems. Generally, the students will try to show, by employing a series of equivalent expressions, that one side of the original proposed identity is equivalent to the other. An alternative is to show that both sides are equivalent to some third expression. The students should be reminded that, since the identity has not yet been proven, they may not multiply both sides by the same expression or add the same terms to each side, etc. An example which shows several techniques is given below:

Example: Show that $\frac{1 - \sin\theta}{\cos\theta} = \frac{\cos\theta}{1 + \sin\theta}$ is an identity

$$\frac{1 - \sin\theta}{\cos\theta} = \frac{\cos\theta}{1 + \sin\theta}$$

$$= \frac{\cos\theta\,(1-\sin\theta)}{(1 + \sin\theta)(1 - \sin\theta)}$$

$$= \frac{\cos\theta\,(1 - \sin\theta)}{1 - \sin^2\theta}$$

$$= \frac{\cos\theta\,(1 - \sin\theta)}{\cos^2\theta}$$

$$= \frac{1 - \sin\theta}{\cos\theta}$$

We conclude that the original open sentence is an identity.

The students should be encouraged to develop such arguments, or equivalent ones, on their own.

B. EQUATIONS

Mathematical techniques involved in equation-solving do not change because of the inclusion of one or more trigonometric functions in the equation. For example, just as the linear equation $2x = 1$ yields the result $x = 1/2$ after division by 2, so does the linear equation $2 \sin x = 1$ yield $\sin x = 1/2$ after division by 2. Just as the left side of the quadratic equation $x^2 - 1 = 0$ can be factored into $(x + 1)(x - 1)$, so can the left side of $\sin^2 x - 1 = 0$ be factored into $(\sin x + 1)(\sin x - 1)$. However, the additional step of finding the required angle, or angle-measure, after solving for the trigonometric function of that angle, will be a new idea for most of the students.

Equations containing trigonometric functions can be categorized, like others, according to degree, in this case, the degree to which the trigonometric function is raised. In Course III, the students will encounter linear and quadratic trigonometric equations. After the students find the function value, the solution set may include one or more of the special angles (30°, 180°, 225°, etc.) and/or approximate measures obtained from a table or calculator. It is necessary to note carefully the domain of the variable. Sometimes it will be specified as an interval, for example, 0° to 360° (0 to 2π radians). In other problems, all angles for which the indicated open sentence is true will be required.

1. Linear Equations

First degree trigonometric equations can be solved using the methods the students have used for other linear equations.

Example 1:

Solve: $2 \sin \theta - 1 = 0$ for the domains

i. $0° \le \theta \le 360°$

ii. all values of θ

Solution: $2 \sin \theta - 1 = 0$

$2 \sin \theta = 1$

$\sin \theta = 1/2$

For (i), $\theta = 30°, 150°$

For (ii), $\theta = 30° + (360n)°, 150° + (360n)°$, n an integer

Of course, answers may also be given in terms of radians; for (ii) this would be $\pi/6 + 2\pi n, 5\pi/6 + 2\pi n$, n an integer.

Frequently a sketch of the function involved is helpful, as the following example illustrates. The sketch reminds us of the sign of the function in the various quadrants, and we can also make use of whatever symmetries the function possesses.

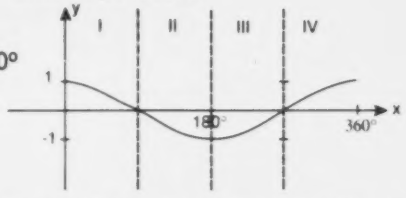
Example 2:

Solve: $4 \cos \theta = \cos \theta + 2$ for $0° \le \theta < 360°$

$4 \cos \theta = \cos \theta + 2$

$3 \cos \theta = 2$

$\cos \theta = 2/3$



A table or calculator yields $\theta \approx 42°$ to the nearest degree. Since the cosine is positive in Quadrants I and IV, as shown by the sketch, the final answer is $\theta \approx 42°, 318°$.

2. *Quadratic Equations*

For quadratic trigonometric equations, it is the trigonometric function which is raised to the second degree. As with quadratic equations the students have met before, some may be solvable by inspection or factoring; others will require the quadratic formula.

Example 1:

Find all values of θ in the interval $0° \le \theta < 360°$ which satisfy $\cos^2 \theta - 1 = 0$.

Solution: Treat $\cos \theta$ as the variable and factor. This yields $\cos \theta = \pm 1$. The roots are $0°$ and $180°$.

Sometimes correct equation-solving techniques lead to a value of a trigonometric function which is impossible, as the following example illustrates.

Example 2:

Find all values of θ in $0° \le \theta < 360°$ which satisfy:

$2 \cos^2 \theta + 7 \cos \theta - 4 = 0$

Solution: $2 \cos^2 \theta + 7 \cos \theta - 4 = 0$

$(2 \cos \theta - 1)(\cos \theta + 4) = 0$

$\cos \theta = 1/2$ $\quad$ $\cos \theta = -4$

$\theta = 60°, 300°$ $\quad$ No solution. (Cos θ cannot be less than -1.)

The following example requires the quadratic formula as well as one of the Pythagorean Identities to rewrite the equation in terms of a single variable. Also, a table (or a calculator) will be necessary to find the final answer.

Example 3:

Solve for x in $0° \le x < 360°$: $2\sec^2 x - 3\tan x - 5 = 0$

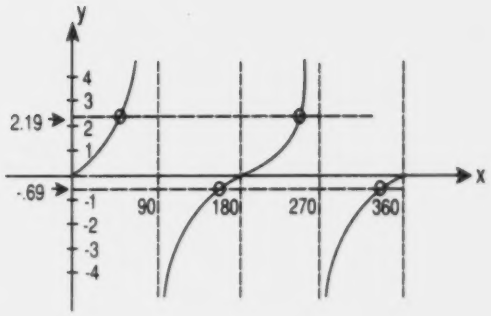
Solution: Use the identity $\sec^2 x = \tan^2 x + 1$.

$$2(\tan^2 x + 1) - 3\tan x - 5 = 0$$
$$2\tan^2 x + 2 - 3\tan x - 5 = 0$$
$$2\tan^2 x - 3\tan x - 3 = 0$$

Then treat tan x as the variable and use the quadratic formula.

$$\tan x = \frac{3 \pm \sqrt{33}}{4} \approx 2.19 \text{ or } -0.69$$

Since tan x may take on any real value, each of these results yields two roots: $x \approx 65°, 145°, 245°, 325°$. These roots are shown in the sketch at the left.



C. FUNCTIONS OF SUMS AND DIFFERENCES OF ANGLES

1. Sine, Cosine, Tangent

The students should now be ready to appreciate the derivation of the formulas for the sine, cosine, and tangent of the sum or difference of two angles:

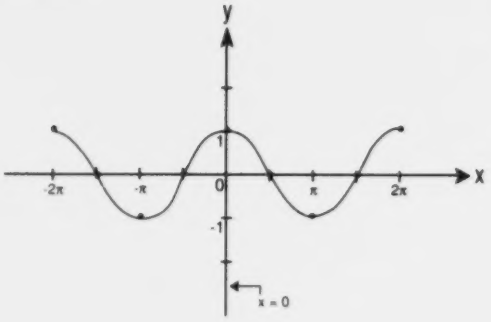
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \quad \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

The usual procedure is to derive the formula for cos (x - y) first and then use this formula to derive the others. The derivation for cos (x - y) is lengthy, and the teacher is referred to a textbook for such a presentation. Although the students will not be asked to derive these formulas on the Regents examination, their derivations should be presented in class.

Because of their similarity, these formulas are sometimes difficult for the students to remember accurately. The teacher should not emphasize the memorization of the formulas, but concentrate instead on their derivation and their place in the structure of trigonometry. The formulas will be provided on the Regents examination.

Example 1:

The sketch at the right shows the graph of y = cos x symmetric with respect to the line x = 0. That is, it appears that cos x is equal to cos (-x) for all x. Show analytically that this is true.

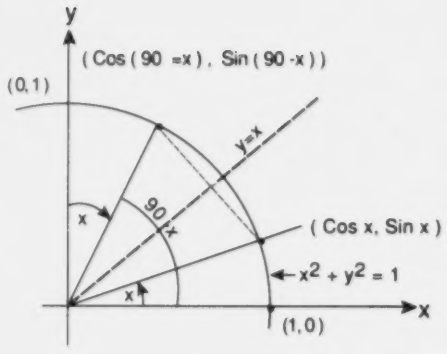


Solution: Use the formula for cos (x - y).

$$\cos(-x) = \cos(0 - x)$$
$$= \cos 0 \cos x + \sin 0 \sin x$$
$$= 1 \cdot \cos x + 0 \cdot \sin x$$
$$= \cos x$$

Example 2:

The sketch at the right, interpreted in light of transformation geometry—in this case, the fact that the image of a point (x,y) in the line y = x is (y,x)—shows that sin x = cos (90° - x) for any acute angle x. A demonstration similar to that of Example 1 will show the statement true for all x.



Many other relationships like the one discussed in Example 2 can be investigated by using geometric and/or analytic means. In particular, we can prove the identity sin (-x) = -sin x, which is used in the proof of the formula for cos (x + y).

Example 3:

Show that cos (x + y) = cos x cos y - sin x sin y.

Solution: Use the identities cos (-x) = cos x and sin (-x) = -sin x to express cos (x + y) as cos [x - (-y)].

The formula for tan (x + y) can be derived by expressing tan (x + y) as $\frac{\sin (x + y)}{\cos (x + y)}$, using the formulas for those expressions, and dividing numerator and denominator of the resulting quotient by cos x cos y. The formula for tan (x - y) is derived in a similar way.

2. Applications

Conceptually, probably the most important application of any proven statement is its use in the derivation of additional facts. But the formulas discussed here have applications in the evaluation of various trigonometric functions which are important in them-

selves, and which can give the students useful arithmetic and algebraic practice. For example, using the formula for tan (x + y) can afford practice in handling complex fractions.

Example 1:

Find the exact value of tan 15°

Solution: $\tan 15^\circ = \tan(45^\circ - 30^\circ)$

$$= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}} \cdot \frac{\frac{3}{1}}{\frac{3}{1}} = \frac{3 - \sqrt{3}}{3 + \sqrt{3}}$$

Example 2:

Suppose that w is an angle in Quadrant I and t is an angle in Quadrant II. If sin w = 2/3 and cos t = -4/5, find the value of sin (w + t).

Solution: Use the fact that $\sin^2 x + \cos^2 x = 1$ for any angle x to find values for cos w and sin t. Once this is done, the formula for sin (x + y) can be used to complete the problem.

D. DOUBLE AND HALF ANGLE FORMULAS

Since these formulas will be provided on the Regents examination, emphasis should be placed upon their derivations (not required for the Regents) and upon their applications.

1. Double Angle Formulas

The formulas for the sine, cosine, and tangent of the sum of two angles can be used to derive formulas for sin 2x, cos 2x, and tan 2x, respectively.

$\sin 2x = \sin(x + x)$

$= \sin x \cos x + \cos x \sin x$

$= 2 \sin x \cos x$

$\cos 2x = \cos(x + x)$

$= \cos x \cos x - \sin x \sin x$

$= \cos^2 x - \sin^2 x$

$$\tan 2x = \frac{\tan x + \tan x}{1 - \tan x \tan x}$$

$$= \frac{2 \tan x}{1 - \tan^2 x}$$

The Pythagorean Identity $\sin^2 x + \cos^2 x = 1$ can be used to derive the following alternate formulas for cos 2x:

$\cos 2x = 2 \cos^2 x - 1$ $\qquad$ $\cos 2x = 1 - 2 \sin^2 x$

Example: If y is an angle in Quadrant II and cos y = -4/5, find sin 2y.

Solution: First it is necessary to find sin y. Then the formula for sin 2x is used.

$\sin^2 y + \cos^2 y = 1$

$\sin^2 y + \frac{16}{25} = 1$

$\sin y = \pm \frac{3}{5}$

Since y is in Quadrant II, $\sin y = +\frac{3}{5}$.

$$\sin 2y = 2 \sin y \cos y$$

$$= 2 \cdot \frac{3}{5} \cdot \frac{-4}{5} = \frac{-24}{25}$$

The sign of this result is at least reasonable since if y is in Quadrant II, 2y is in Quadrant III or IV, where sine is negative.

Example 2:

Find tan 90° by using the formula for tan 2x.

Solution: $\tan 90° = \tan (2 \cdot 45°)$

$$= \frac{2 \tan 45°}{1 - \tan^2 45°}$$

$$= \frac{2 \cdot 1}{1 - 1 \cdot 1}$$

But this fraction, 2/0, is meaningless. Hence tan 90° does not exist.

2. *Half Angle Formulas*

The Double Angle formulas can be used to derive the formulas for the sine and cosine of one-half an angle:

$$\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$$

The formula for the tangent of 1/2 x can be found by noting that

$\tan \frac{1}{2} x = \frac{\sin \frac{1}{2} x}{\cos \frac{1}{2} x}$ and by using the two formulas shown above.

This yields

$$\tan \frac{1}{2} x = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

Note that each of the half angle formulas provides for a positive or negative result, depending on the conditions of each case. Care must be taken to choose the correct sign.

Example: Find the exact value of cos 195°.

Solution: $\cos 195° = \cos \frac{390°}{2} = \pm \sqrt{\frac{1 + \cos 390°}{2}}$

$$= \pm \sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}} = \pm \sqrt{\frac{2 + \sqrt{3}}{4}} = \pm \frac{\sqrt{2 + \sqrt{3}}}{2}$$

Since 195° is in Quadrant III, cos 195° is negative. Thus, $\cos 195° = -\frac{1}{2} \sqrt{2 + \sqrt{3}}$.

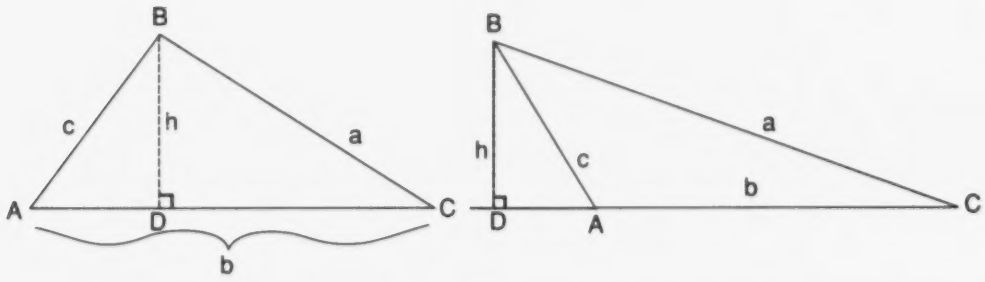
VII.
Trigonometry – Triangle Solution

In Course II, the students had a brief introduction to the trigonometry of the right triangle. In Course III, the approach to the study of trigonometry is through the general angle. Triangle trigonometry investigates the relationships between the sides and angles of any triangle, and relates these to the right triangle trigonometry studied in Course II. The emphasis in this section should be on problem solving.

A. AREA OF A TRIANGLE

The formula for the area of a triangle which the students have used in Courses I and II can now be extended using the sine function. The formula for the area using any two sides of the triangle and the sine of the included angle, $A = \frac{1}{2}ab \sin C$ should be derived with the students, but the emphasis should be on its application.

In either acute or obtuse ΔABC, the area can be represented as $A = \frac{1}{2}bh$.



Since $\sin A = \frac{h}{c}$, $h = c \sin A$. Thus the area of either triangle can be expressed as $A = \frac{1}{2}bc \sin A$. Using side a as a base, it can similarly be shown that $A = \frac{1}{2}ac \sin B$ and using side c as a base, $A = \frac{1}{2}ab \sin C$. Thus the area of a triangle can be expressed as one-half of the product of any two sides and the sine of the included angle.

B. GENERAL SOLUTION OF TRIANGLES

The Law of Sines and Law of Cosines provide two important general relationships between the sides and angles of *any* triangle—acute, right, or obtuse. Derivations of these two formulas should be presented, but on the Regents examination the students will not be held responsible for their derivations. The applications of the Laws of Sines and Cosines should be stressed.

1. Law of Sines

A common derivation for the Law of Sines uses expressions for the area of a triangle. In either acute or obtuse Δ ABC, the area can be represented by $A = \frac{1}{2}$ bc sin A, $\frac{1}{2}$ ac sin B, or $\frac{1}{2}$ ab sin C.

Thus $\frac{1}{2}$ bc sin A = $\frac{1}{2}$ ac sin B = $\frac{1}{2}$ ab sin C

Dividing this expression by 1/2 abc, we can obtain the statement of the Law of Sines. In words, the Law states that in any triangle the ratio between the sine of an angle and the side opposite that angle is a constant.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

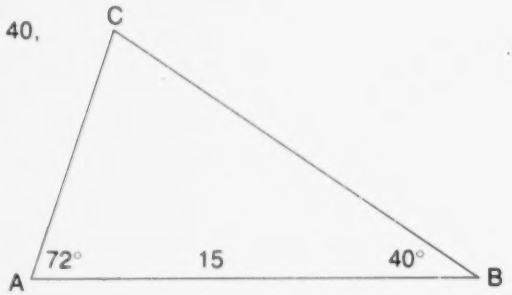
The Law of Sines can be used to find missing parts of a triangle if ASA, AAS, or SSA are the known combinations.

If two angles of a triangle are known, the third angle is easily obtained. Then a proportion using the Law of Sines can be established and the missing side found.

Example:

Given: m $\angle$ A = 72, m $\angle$ B = 40, AB = 15

Find: BC to the nearest tenth



Solution: m $\angle$ C = 68

Using $\frac{\sin C}{c} = \frac{\sin A}{a}$,

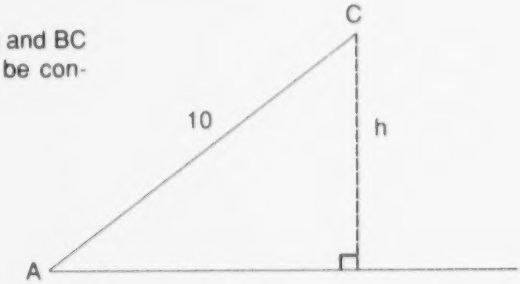
$$\frac{\sin 68^\circ}{15} = \frac{\sin 72^\circ}{a}$$

$$a = 15.4$$

The students should know that two triangles are not necessarily congruent if two sides and the angle opposite one of them in one triangle are equal to the corresponding sides and angle in the other. Using ideas studied in trigonometry, they can explore the reason why such triangles are not necessarily congruent. Several examples should be given with particular values which show that either 0, 1 or 2 triangles are actually possible. This SSA combination is known as the Ambiguous Case.

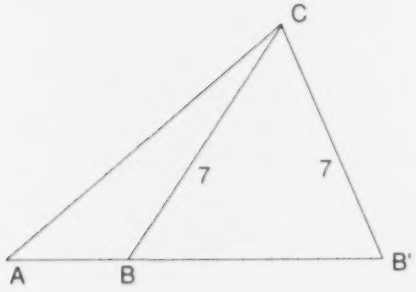
Example:

In Δ ABC, m $\angle$ A = 40, AC = 10 and BC = 7. How many triangles can be constructed?



Solution: h = 10 sin 40°

h $\approx$ 6.4



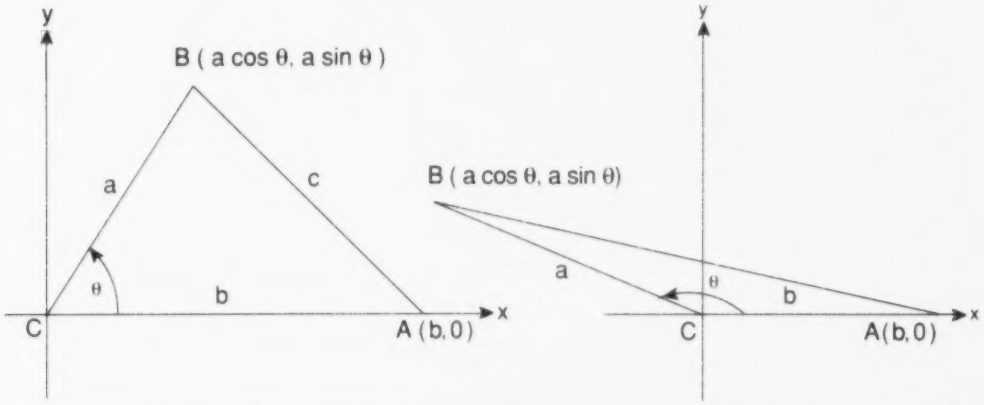
Since 6.4 < BC < 10, there are two possible triangles, Δ ABC and Δ AB'C.

The Law of Sines can be used to solve for the missing parts of a [illegible] en SSA) after it is determined how many triangles are possible. If two tri[illegible] be formed, there will be two sets of solutions.

2. Law of Cosines

One method which can be used for the derivation of the Law of Cosines uses the distance formula for the distance between two points.

The coordinates of A and B in either the acute or the obtuse triangle are (b,0) and (a cos θ, a sin θ), respectively.



Using the distance formula to express the distance between A and B, we have the following:

$$c = \sqrt{(b - a\cos\theta)^2 + (0 - a\sin\theta)^2}$$

$$c^2 = b^2 - 2ab\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta$$

$$c^2 = b^2 - 2ab\cos\theta + a^2(\cos^2\theta + \sin^2\theta)$$

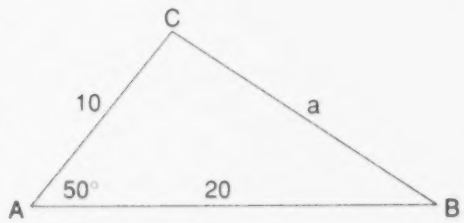
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

This formula expresses one side of a triangle in terms of the other two sides and the included angle. The Law of Cosines can be used to solve problems where SAS or SSS is the given information.

Example 1:

Given: Δ ABC, m $\angle$ A = 50, AC = 10, AB = 20

Find BC.



Solution: Using the Law of Cosines,

$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 100 + 400 - 2(10)(20)\cos 50^0$

Example 2:

Given: Δ PQR, PR = 8, RQ = 17, PQ = 20

Find: m $\angle$ R.

Solution: The appropriate Law of Cosines can either be written as

$$r^2 = p^2 + q^2 - 2pq\cos R$$

or $$\cos R = \frac{p^2 + q^2 - r^2}{2pq}$$

3. Applications

The Law of Sines and the Law of Cosines should be applied to a wide variety of problems involving triangles, parallelograms, and other geometric figures. Applications involving the resolution of forces should be included. Note the similarity between the graphic representation of the sum of two forces and the sum of two complex numbers.

Example 1:

Main Street and Park Avenue intersect at an angle of 74°. Mr. Jones lives on Main Street, 50 meters from the intersection, and Mr. Smith lives on Park Avenue, 40 meters from the intersection. Find, to the nearest meter, the distance between Mr. Jones' house and Mr. Smith's house.

Example 2:

Two forces act on a body at an angle of 72°, producing a force whose magnitude is 110 lb. If the magnitude of one force is 80 lb., find the magnitude of the other force.

Example 3:

The angle of elevation from a ship at point A to the top of a lighthouse at point B is 43°. The ship moves in a straight line toward the foot of the lighthouse. When the ship reaches point C, 300 meters closer to the lighthouse, the angle of elevation is 56°. Find, to the nearest meter, the height of the lighthouse.

C. RIGHT TRIANGLE TRIGONOMETRY

1. Definitions

In Course II, sin A, cos A, and tan A were expressed in terms of the sides of a right triangle. These relationships should now be compared with the definitions of the functions of the general angle. It should be pointed out that the right triangle ratios are a special case of the general angle functions.

If in Δ ABC, $\angle$ C is a right angle, then $\frac{a}{\sin A} = \frac{c}{\sin 90°}$

or $\sin A = \frac{a}{c} = \frac{\text{leg opposite } \angle A}{\text{hypotenuse}}$

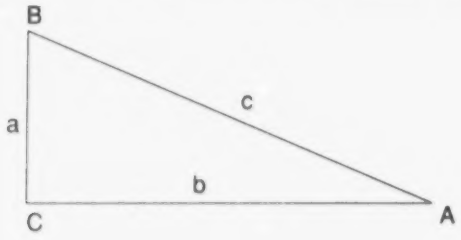
In this same triangle, $a^2 = b^2 + c^2 - 2bc \cos A$.

Since $c^2 = a^2 + b^2$, $c^2 - b^2 = b^2 + c^2 - 2bc \cos A$

or $\frac{b}{c} = \cos A = \frac{\text{leg adjacent } \angle A}{\text{hypotenuse}}$

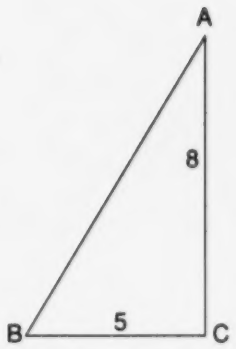
Using the identity, $\tan A = \frac{\sin A}{\cos A}$, $\tan A = \frac{a}{b} = \frac{\text{leg opposite } \angle A}{\text{leg adjacent } \angle A}$

Using the right triangle ratios, the students should be able to find, to the nearest 10 minutes, either acute angle of a right triangle if they are given any two sides.



Example:

Given Δ ABC with right $\angle$ C,
AC = 8 and BC = 5. Find m $\angle$ A.



Solution: $\tan A = \frac{BC}{AC}$

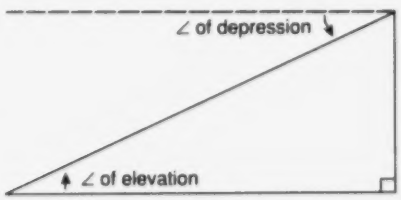
$\tan A = \frac{5}{8}$

$\tan A = 0.6250$

$A \approx 32^\circ$

2. *Applications*

The concepts of the angles of elevation and depression should be reviewed and the relationship between them noted.



There are many problems whose solutions depend upon the application of one of the trigonometric functions in a right triangle.

Example 1:

Find, to the nearest foot, the height of a tree if the angle of elevation of the top of the tree from an observer 50 feet away from the tree is 18°.

Example 2:

An ideal wheelchair ramp makes an angle of 23° with the horizontal. If the doorway is located three feet above ground level, what would be the length of the ramp, to the nearest foot? How far from the building would the ramp start?

Example 3:

One angle of a rhombus is 76°. The shorter diagonal has a length of 18 cm. Find to the nearest centimeter the length of the side of the rhombus.

Example 4:

In isosceles trapezoid ABCD, the bases are 10 and 16 and m $\angle$ A = 70. Find, to the nearest unit, the length of leg $\overline{AD}$.

VIII.
Probability

The probability strand in this Course is a continuation of the probability strand in Course I and Course II. In those courses, the students learned basic ideas about finding the probability of events and about combinatorics. Permutations were studied in Course I, and combinations introduced in Course II. In Course III, work with combinations is extended through one important application: finding the probability of exactly r successes of a Bernoulli experiment. In addition, the Binomial Theorem is introduced.

The section begins with a review of combinations, but the teacher may also wish to review permutations and some fundamental concepts of probability.

A. REVIEW OF WORK WITH COMBINATIONS

The students should review what a combination is and how it differs from a permutation. They should review and practice the methods for finding the number of combinations of n things taken r at a time, and should understand the use of the following formulas:

$$_nC_r = \frac{_nP_r}{r!} \qquad _nC_r = \frac{n!}{r!(n-r)!}$$

$$_nC_r = \frac{n(n-1)(n-2)(n-3)\ldots(n-r+1)}{r!}$$

Examples:

1. Discuss whether each of the following is a permutation problem or a combination problem:
   a. Finding the number of ways a president and a secretary can be chosen from a club of 15 members
   b. Finding the number of ways a subcommittee of four can be selected from a committee of 24
   c. Finding the number of different hands that can initially be dealt in 5-card draw poker
   d. Finding the number of "words" which can be formed from the letters in TOADS (no repeats allowed).
2. In how many ways can a subcommittee of four people be chosen from a committee of ten if one of the ten, Jones, must be among the four?
3. A committee has four Republicans and seven Democrats. A subcommittee of four is chosen by lot. What is the probability that the subcommittee will have two Republicans and two Democrats?

Solution:

A "successful" subcommittee can be chosen in any one of ${}_4C_2 \cdot {}_7C_2 = 6 \cdot 21 = 126$ ways. The total number of ways a subcommittee can be chosen is ${}_{11}C_4 = \frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{7920}{24} = 330$.

Thus the required probability is $\frac{126}{330}$.

B. PROBABILITY OF EXACTLY R SUCCESSES IN N TRIALS OF A BERNOULLI EXPERIMENT

1. Definition of a Bernoulli Experiment

A Bernoulli experiment is one in which exactly two distinct outcomes are identified. Examples are:

i) tossing a fair die; outcome odd or even

ii) tossing a fair die; outcome 5 or not 5

iii) tossing an unfair die; either of the outcome pairs given in (i) or (ii)

iv) shooting at a target; hitting or not hitting the target

v) teams A and B playing a certain game; team B wins or does not win.

Usually an occurrence of one of the pair of outcomes is specified as a success; each instance, or toss, etc., of the experiment is called a trial. Since the two outcomes constitute the entire outcome set, the sum of their probabilities must be 1. If the probability of a success in any one trial is p, then the probability of failure, q, must be equal to 1 - p.

The problem of finding the probability of exactly r successes in n trials of a Bernoulli experiment can appear difficult for some students. A good approach to deriving the required rule is to break the problem up into pedagogical steps. We will refer to the following problem:

What is the probability of rolling exactly one 6 in six tosses of a fair die?

Before proceeding, the nature of the problem should be discussed. A common error is to believe that since there are six possible outcomes, and each has a probability $\frac{1}{6}$, the event described is certainty or near-certainty. But that is a hypothesis which can easily be tested empirically, and such demonstrations are recommended.

2. Case Where r Successes Are Assumed to Occur First

Refer to the problem stated above. If the 6 is required to occur first, followed by five non-6's, the probability of that sequence of trials is:

$$\frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^5$$

3. General Case

In our original problem, the 6 was not required to occur first. In fact, it can occur in any one of the six positions (tosses); thus, to find the probability asked for in the original problem, we must multiply the result obtained above by 6. Hence,

P(exactly one 6 in six tosses) =

$$6 \cdot \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^5 = \frac{3125}{7776} \approx 0.402$$

If it is required to find the probability of obtaining two 6's and four non-6's, in that order, then the probability of that sequence is:

$$\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6} = \left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4$$

For the probability of obtaining two 6's in six tosses in *any* order—6's not necessarily first—it is clear that the two 6's can be obtained on any two of the six available tosses. This selection can occur in any of ${}_6C_2$ ways. Thus,

P(exactly two 6's in six tosses) =

$${}_6C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4 = 15\cdot\frac{1}{36}\cdot\frac{625}{1296} = 0.201$$

It follows that the probability of obtaining exactly r 6's in six tosses ($0 \le r \le 6$) is given by

$${}_6C_r\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{6-r}$$

and that the probability of obtaining exactly r 6's in n tosses ($0 \le r \le n$) is

$${}_nC_r\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{n-r}$$

Finally, the generalization for any Bernoulli experiment can be made:

Given a Bernoulli experiment with the probability of success p on each trial and the probability q = 1 - p of failure, then the probability of obtaining exactly r successes in n trials of this experiment is

$${}_nC_r\, p^r\, q^{n-r}$$

4. *Applications*

Some accessible applications of the rule derived in this section are indicated in the preceding discussion. Here are some additional examples.

1. A coin has been loaded so that P(heads) = $\frac{2}{3}$. What is the probability of obtaining exactly three heads in four tosses of this coin?

   Solution: Consider heads to be a success. Then, $p = \frac{2}{3}$, $q = \frac{1}{3}$, r = 3 and n = 4.

   We have:

   $$\text{P(exactly 3 Heads)} = {}_4C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^1$$

   $$= 4\cdot\frac{8}{27}\cdot\frac{1}{3} = \frac{32}{81}$$

2. What is the probability of obtaining exactly 500 heads in 1000 tosses of a fair coin?

   Solution: $p = \frac{1}{2}$, $q = \frac{1}{2}$, r = 500, n = 1000

   $$\text{P(500 heads)} = {}_{1000}C_{500}\left(\frac{1}{2}\right)^{500}\left(\frac{1}{2}\right)^{500} = {}_{1000}C_{50}\left(\frac{1}{2}\right)^{1000}$$

   Note that this expression is virtually impossible to evaluate exactly.

3. What is the probability of obtaining *at least* eight heads in ten tosses of a fair coin?

   Solution: "At least eight" means "exactly eight or exactly nine or exactly ten." Thus P(at least 8) = P(8) + P(9) + P(10).

   P(at least eight heads) =

$$ {}_{10}C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + {}_{10}C_9\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^1 + {}_{10}C_{10}\left(\frac{1}{2}\right)^{10} $$

C. THE BINOMIAL THEOREM

Many students may have seen the binomial coefficients in their work with algebra. In any case, the students can easily verify that the expansion of $(p + q)^n$ for n = 1,2,3,4,...will yield coefficients shown below on the right. This is the arrangement of numbers known as Pascal's Triangle.

$(p+q)^0 = 1$	1
$(p+q)^1 = p+q$	1 1
$(p+q)^2 = p^2 + 2pq + q^2$	1 2 1
$(p+q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$	1 3 3 1
$(p+q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$	1 4 6 4 1
$(p+q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$	1 5 10 10 5 1

The students' work with expansion of binomials and their knowledge of combinatorics may lead to a statement of the Binomial Theorem:

$$(p + q)^n = {}_nC_0p^n + {}_nC_1\,p^{n-1}q^1 + \ldots + {}_nC_nq^n$$

The students should practice with the Theorem, and they should be able to handle simple applications.

Examples

1. Write the expansion of $(x - 2y)^6$.
2. Write the second term of the expansion of $(3a - 1)^4$.
3. Write an expression (do not simplify it) for the next-to-last term of the expansion of $\left(z^2 + \frac{1}{3}\right)^{30}$.

IX.
Statistics

In Course III, the students will review the statistical ideas from Course I and continue with a study of measures of dispersion. The work culminates with a discussion of the standard deviation, perhaps the most important single statistical measure both in its own right and as the foundation for more advanced work with statistical inference.

As in Course I, the teacher should devote some time to the students' discussion of the social and ethical aspects of today's large-scale reliance on statistical inference.

A. REVIEW OF MEASURES OF CENTRAL TENDENCY

In Course I, the students studied the three best known measures of central tendency: the mean (arithmetic mean or average), the median, and the mode. These are measures of central tendency, or measures about which a collection of measures tend in some way to cluster. The students should review their understanding of these measures and the distinctions among them as a prelude to future work in this unit.

Examples:

1. For each of the following sets of measures, i) find the mean, the median, and the mode, and ii) discuss the degree to which each of these represents the overall nature of the indicated set.
   a. { 2, 3, 4, 4, 4, 6, 6, 8 }
   b. { 2, 3, 10, 12, 14, 14, 21, 500 }
2. The arithmetic mean of the numbers 5, 7, $x + 3$, and 12 is 10. What is the value of x?

B. Σ-NOTATION

The introduction of summation notation could take place naturally at several points in the three-year sequence. However, its introduction at this point, in conjunction with the ideas of statistics, where this notation is so frequently and necessarily used, seems justified, especially in terms of providing reasonable, nonartificial applications.

The summation symbol Σ (the Greek capital letter sigma) means "take the sum of what follows" and is generally used with some indexing (or "dummy") letters i, j, k, etc. to indicate the limits of summation. For example,

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

The expression on the left of the equals sign means the sum of the integers designated by i, from $i = 1$ to 5. The lower limit of summation is 1, and the upper limit is 5.

The final sum is obtained by substituting into the expression immediately to the right of the Σ each of the integers from the lower to the upper limits of summation, inclusive, and adding all terms thus obtained. Any letter could have been used for i; the meaning of the expression would be unchanged. Sometimes, if the written limits of summation are clear from the context, they may be omitted.

Examples:

Evaluate each of the following

1. $\sum_{i=0}^{4} (i^2 + 1)$

2. $\sum_{k=2}^{5} \frac{k^3}{4}$

If each of a set of numbers is denoted by a general symbol, for example x, we can *index* them for future reference: $\{ x_1, x_2, x_3, \ldots x_i \}$. If it is known that there are n such measures, the last term in the set could be denoted xn. This technique of indexing, together with summation notation, can be used to work with sets of numbers. For example, the mean of the set of numbers $\{ x_1, x_2, x_3, \ldots x_n \}$ is

$$\frac{1}{n} \sum_{i=1}^{n} x_i$$

The mean of $\{ x_1, x_2, x_3, \ldots, x_n \}$ is usually symbolized $\bar{x}$, read "x-bar."

The students should gain facility with the use of Σ -notation.

C. MEASURES OF DISPERSION

Although measures of central tendency are very widely used and obviously very important, for certain kinds of investigation they are insufficient. For example, a mean score of 80 percent on the field test of a standardized examination could represent a majority of scores between 75 percent and 85 percent, or it could represent about half with 60 percent and half with 100 percent—clearly two very different results. What is needed in addition to measures which describe how a set of numbers cluster around a central number is measures which describe the extent to which the numbers in the set are scattered, or dispersed, from that central number. Such measures are called measures of dispersion.

1. Range

Probably the simplest measure of dispersion is the *range*, which is the difference between the greatest and the least numbers in the set. Sometimes this measure is useful, but since it depends on only two numbers, one or both of which may not be representative, it is easy to see circumstances in which this measure would be inadequate. (For instance, see Example 1 under subsection A of this section.)

2. Mean Absolute Deviation

Some students might be tempted to suggest, as an acceptable measure of dispersion which makes use of all the numbers in the set, the sum of all their deviations from the mean, in symbols,

$$\sum_{i=1}^{n} (x_i - \bar{x}).$$

Unfortunately this sum always turns out to be zero; an investigation of this fact is a good classroom activity. However, if the sign of each difference $(x_i - \bar{x})$ is disregarded, then each of the differences would become nonnegative and their sum positive (except in the rare instance when all the numbers in the set were equal). A measure of dispersion based on this idea is called the *mean absolute deviation*, in symbols,

$$\frac{1}{n}\sum_{i=1}^{n} |x_i - \bar{x}|$$

3. Variance

A more widely used measure of dispersion is the *variance*. With this measure the deviations from the mean are squared, which assures that they will be nonnegative. The variance is the sum of these squared deviations, divided by the number of measures. In symbols, for n measures:

$$var = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

Another symbol for the variance, besides *var*, is s^2. The reason for this notation will become clear in the discussion of standard deviation. Also,that discussion includes an example of finding the variance of a set of numbers.

4. Standard Deviation

Two possible disadvantages of the variance are the apparent distortion introduced by the squaring of each difference from the mean, and the relatively large size of the measure compared to the set of numbers being described. These objections are answered at least partially by taking *the square root of the variance*. The square root of the variance is the *standard deviation*. In symbols, for n measures:

$$S.D. = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

Other symbols for the standard deviation are s and σ (Greek lower case sigma).

Example:

Find the variance and standard deviation of the following set of measurements: { 4, 8, 9, 10, 10, 11, 12, 12, 12, 13, 15, 16.}.

Solution: First, we note that n = 12. Then we find the mean. We compute $\bar{x} = 11$.

In this example much of the remaining computation is shown in table form. This is frequently useful, but, of course, no particular arrangement of information is required.

index(i)	measure(x_i)	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
1	4	-7	49
2	8	-3	9
3	9	-2	4
4	10	-1	1
5	10	-1	1
6	11	0	0
7	12	1	1
8	12	1	1
9	12	1	1
10	13	2	4
11	15	4	16
12	16	5	25

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 = 112$$

$$s^2 = \text{var} = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{112}{12} = 9.\overline{3}$$

$$s = \text{S.D.} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2} = \sqrt{9.3} \approx 3.06$$

The computation of variance and standard deviation as defined requires the evaluation of $x_i - \bar{x}$ for all measures and the squaring of those differences. The identity shown below, however, can sometimes simplify computation, and more able students can be encouraged to derive it (The first step is to expand the expression $(x_i - \bar{x})^2$).

$$\frac{1}{n}\sum_{i=1}^{n} (x_i - x)^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

Sometimes a set of measurements consists of a few numbers occurring repeatedly. Then we may *group* the measures, and index them, according to their *frequency* (denoted by f).

Example:

Find the standard deviation of the following set of measures: { 7, 1, 7, 3, 10, 7, 0, 1, 3, 6, 4, 0, 7, 7, 3 }.

Solution: It is useful to order the measures first:

0, 0, 1, 1, 3, 3, 3, 4, 6, 7, 7, 7, 7, 7, 10

Calculating the mean, we have

$$\bar{x} = \Sigma f_i x_i \div f_i$$
$$= [2(0) + 2(1) + 3(3) + 1(4) + 1(6) + 5(7) + 1(10)] \div 15 = 66/15 = 4.4$$

For this data, then, the standard deviation is given by

$$\text{S.D.} = \sqrt{\frac{1}{15}\sum f_i (x_i - \bar{x})^2}$$

This problem is completed with the aid of the following table.

meas. (x_i)	freq. (f_i)	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$	fi $(x_i - \bar{x})^2$
0	2	-4.4	19.36	38.72
1	2	-2.4	5.76	11.52
3	3	-1.4	1.96	3.92
4	1	-0.4	0.16	0.16
6	1	1.6	2.56	2.56
7	5	2.6	6.76	33.80
10	1	5.6	31.36	31.36

$$\text{var} = \frac{1}{15}\sum_{i=1}^{7} f_i (x_i - \bar{x})^2 = 122.04 \div 15 = 8.136$$

$$\text{S.D.} = \sqrt{\text{var}} = \sqrt{8.136} \approx 2.85$$

In general, for a set of n measures, with k different, an expression for the standard deviation is

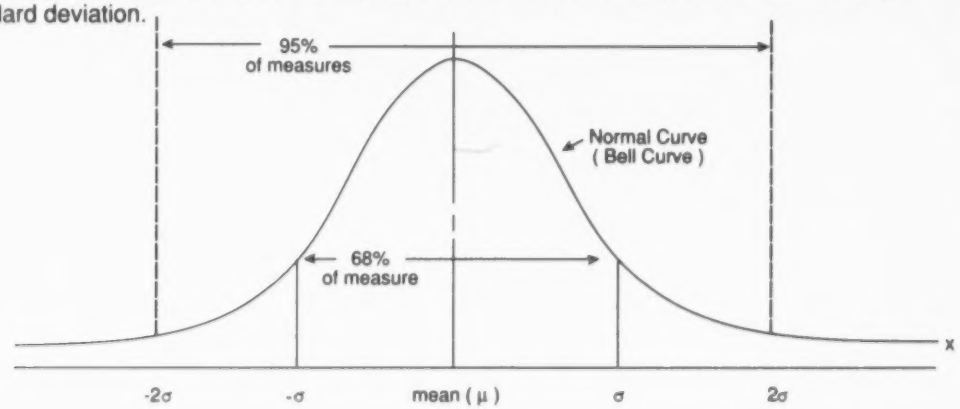
$$\sqrt{\frac{1}{n}\sum_{i=1}^{k} f_i (x_i - \bar{x})^2}$$

Here, n could be replaced by $\sum_{i=1}^{k} f_i$.

D. BELL CURVES

A frequently-used application of the standard deviation is found in the construction and interpretation of so-called "bell curves." Many sets of measures, especially in the social sciences, occur according to what is called a *normal distribution*. The graph of this distribution, technically a *normal curve*, is frequently referred to as a bell curve. While most of the formal properties of this curve, including its equation, are beyond the scope of this course, it is nonetheless useful for the students to learn about some inferences which can be drawn from a bell curve.

A typical bell curve is shown below. It is symmetric about the line $x = \mu$ where μ represents the mean of a set of measures, and each unit on the horizontal axis is one standard deviation.



It can be shown that when a set of data is distributed normally (that is, when the distribution can be represented graphically by a bell curve), about 68 percent of the data fall within one standard deviation of the mean (i.e., between $\mu - \sigma$ and $\mu + \sigma$), 95 percent within two standard deviations, and 99 percent within three.

Examples:

1. A survey conducted of 15-year-olds in Spruceville showed that the mean number of hours of television watched per week was 20.7. The standard deviation was 3.8. From this information we can conclude that about 68 percent of the population surveyed watched between 16.9 and 24.5 hours per week. Ninety-five percent watched between 13.1 and 28.3 hours. Virtually everyone watched more than 9.3 hours.
2. On a certain standardized examination, the mean was 50 and the standard deviation 3.
   i) About what percent of those taking the exam scored between 44 and 47?
   ii) Judy's score was 59. How did she do, approximately, compared to the rest of the group?

The study of a variety of more advanced statistical techniques and their applications can provide appropriate challenges to more able students. For example, some might be encouraged to investigate the relationship between the normal and the binomial distributions. This relationship illustrates the link between the Probability and Statistics sections in this course.

It is important to note that the illustrative examples in this *Syllabus/Guide*, and in teachers' classrooms, must be limited in complexity—i.e., involve small sets of data—to facilitate instruction. In virtually no "real" application of the statistical ideas being illustrated are the numbers of data so small; usually there are hundreds, or thousands. The students should be made aware of this, especially if they suggest that they are being asked to draw statistical conclusions about sets of data so small that such complicated procedures do not appear needed for a useful study of these sets.

Evaluation

The teacher is encouraged to develop her/his own formative (ongoing) evaluation strategies to monitor the students' achievement throughout the year. These strategies should include written and oral testing, small group work assignments, and whole class projects. Examples presented in this *Syllabus/Guide* may be used as models for assignments or for examination questions.

This Course III mathematics *Syllabus/Guide* is designed for a Regents level course and should be followed by the Three-Year Sequence for High School Mathematics Course III Regents examination.

This *Syllabus/Guide* may be used as the basis of a local Course III for those students pursuing a local diploma. The local course may be less rigorous than the Regents course, and should be followed by a local examination appropriate to the Course as modified by the local district.

Students with handicapping conditions are to be provided with appropriate alternative testing techniques as indicated on their Individualized Education Programs (IEP). Such approved testing modifications should be used consistently in ongoing evaluations required for Regents and local diplomas.

Resources for Teachers

Computers in Mathematics Education. 1906 Association Drive, Reston, VA 22091: National Council of Teachers of Mathematics. 1984 Yearbook. 244 pp.

Answers questions in teachers' minds at all levels, about the use of computers in the classroom. Discusses computing from the standpoints of its challenge, impact, and perspective and as a diagnostic tool, and programming as a means of teaching mathematics. Contains an annotated bibliography.

Dalton, Leroy C. *Algebra in the Real World.* P.O. Box 10888, Palo Alto, CA 94303: Dale Seymour Publications. 1983. pp. 13-18, 123-128.

Provides a collection of mathematical applications that show the beauty of mathematics as well as its applications in the real world. Includes topics such as algebra, geometry, elementary combinatorics, quadratic and exponential relations, rational polynomial expressions, and trigonometry. Contains follow-along worksheets designed to keep the students actively involved in the lessons.

(Joint Committee of the MAA and the NCTM.) *A Sourcebook of Applications of School Mathematics.* 1906 Association Drive, Reston, VA 22091: National Council of Teachers of Mathematics. 1980. 360 pp.

Provides a large number of realistic problems for use in the secondary school classroom. Includes answers and an annotated bibliography.

Maletsky, Evan, ed. and Yunker, Lee E. ed. *NCTM Student Math Notes.* 1906 Association Drive, Reston, VA 22091: National Council of Teachers of Mathematics. 1986. 21 pp.

Designed for use in junior high and senior high school instruction in mathematics. Includes five sets of activities for the students: 1) mathematics on postage stamps; historical procedures and mathematicians, metric conversion, geometric ideas, and formulas; 2) successful simulation, with activities on planning a supermarket, bowling, and random digits, and other suggestions for using a random digit table; 3) proof: geometric, algebraic, and numerical formulas; 4) golden rectangles and ratios with a physical model for generating them, and directions for constructing a golden rectangle and for generating golden ratios via LOGO, and comments on the divine proportion; and, 5) primes, with the Sieve of Eratosthenes, twin primes, and other problems.

Problem Solving in School Mathematics. 1906 Association Drive, Reston, VA 22091: National Council of Teachers of Mathematics. 1980 Yearbook. 256 pp.

Views problem solving as a basic skill that can be learned and must be taught. Contains ideas, problems, examples, and illustrations for classroom teachers at all levels of instruction, and an annotated bibliography.

Resource Problems to Enhance the Teaching of Mathematics. Ewing Hall, Department of Mathematical Sciences, Newark, DE 19716: University of Delaware. 1986. 300 pp.

Provides the teacher with a set of nonstandard problems to be used in presenting in sections or discussions, or to be given to the students as challenging exercises for them to work on. For each problem, identifies a specific subject area and concept(s). Includes problems in arithmetic, algebra, geometry, coordinate geometry, trigonometry, and precalculus.

Sobel, Max A., ed. *Readings for Enrichment in Secondary School Mathematics.* 1906 Association Drive, Reston, VA 22091: National Council of Teachers of Mathematics. 1988. 298 pp.

Composed of articles from the *Mathematics Teacher, Enrichment Mathematics for High School* (NCTM's 28th Yearbook), and *Topics for Mathematics Clubs*, along with three new chapters on the harmonic mean, rotation matrices and complex numbers, and on the use of computers and cal-

culators in performing arithmetic. Offers a variety of opportunities for expanding the mathematics curriculum for academically talented students.

Spacemathematics: A Resource for Secondary School Teachers. United States Government Printing Office, Washington, DC 20402: Superintendent of Documents. 1985. 172 pp.

Designed to serve as a supplement for high school level mathematics. Contains activities to enhance mathematical knowledge and skills and to assist the students in understanding aerospace technology and its achievements. Includes chapters on "Mathematical Aspects of Some Recent NASA Missions," "Computation and Measurement," "Algebra," "Geometry," "Probability and Statistics," "Exponential and Logarithmic Functions," "Trigonometry," "Matrix Algebra," "Conic Sections," and "Calculus." In an appendix, contains an analysis of gravitational forces and the conic section trajectories, and a glossary.

Teaching Statistics and Probability. 1906 Association Drive, Reston, VA 22091: National Council of Teachers of Mathematics. 1981 Yearbook. 256 pp.

Deals with the place of statistics and probability in the curriculum, discusses how people and mathematics cope with uncertainty, and provides ideas for classroom models, strategies, devices, activities, and projects.

The Ideas of Algebra, K-12. 1906 Association Drive, Reston, VA 22091: National Council of Teachers of Mathematics. 1988 Yearbook. 248 pp.

Stresses the idea that algebra is a living, growing curricular field and that we must continue to improve our presentation so that every youngster will experience the thrill of understanding. Some areas of emphasis: the accessibility of algebraic ideas to young learners before the formal introduction of algebra, equations and expressions, word problems, and the use of technology in the algebra classroom. In last section, focuses on helpful, teacher-tested ideas for teaching algebra.

Appendix A

STATEMENT OF REGENTS GOALS FOR ELEMENTARY AND SECONDARY SCHOOL STUDENTS - 1984

1. Each student will master communication and computation skills as a foundation to:
   - 1.1 Think logically and creatively.
   - 1.2 Apply reasoning skills to issues and problems.
   - 1.3 Comprehend written, spoken and visual presentations in various media.
   - 1.4 Speak, listen to, read, and write clearly and effectively in English.
   - 1.5 Perform basic mathematical calculations.
   - 1.6 Speak, listen to, read, and write at least one language other than English.
   - 1.7 Use current and developing technologies for academic and occupational pursuits.
   - 1.8 Determine what information is needed for particular purposes and be able to acquire, organize and use that information for those purposes.
2. Each student will learn methods of inquiry and knowledge gained through the following disciplines and use the methods and knowledge in interdisciplinary applications:
   - 2.1. English language and literature.
   - 2.2 History and social science.
   - 2.3 Mathematics.
   - 2.4 Natural sciences and technology.
   - 2.5 Language and literature in at least one language other than English.
3. Each student will acquire knowledge, understanding, and appreciation of the artistic, cultural, and intellectual accomplishments of civilization and develop the skills to express personal artistic talents. Areas include:
   - 3.1 Ways to develop knowledge and appreciation of the arts.
   - 3.2 Aesthetic judgments and the ability to apply them to works of art.
   - 3.3 Ability to use cultural resources of museums, libraries, theaters, historic sites, and performing arts groups.
   - 3.4 Ability to produce or perform works in at least one major art form.
   - 3.5 Materials, media and history of major art forms.
   - 3.6 Understanding of the diversity of cultural heritages.
4. Each student will acquire knowledge about political, economic, and social institutions and procedures in this country and other countries. Included are:
   - 4.1 Knowledge of American political, economic and social processes and policies at national, state, and local levels.
   - 4.2 Knowledge of political, economic, and social institutions and procedures in various nations; ability to compare the operation of such institutions; and understanding of the international interdependence of political, economic, social, cultural and environmental systems.
5. Each student will respect and practice basic civic values and acquire the skills, knowledge, understanding, and attitudes necessary to participate in democratic self-government. Included are:
   - 5.1 Understanding and acceptance of the values of justice, honesty, self-discipline, due process, equality, and majority rule with respect for minority rights.
   - 5.2 Respect for self, others and property as integral to a self-governing, democratic society.
   - 5.3 Ability to apply reasoning skills and process of democratic government to resolve societal problems and disputes.

6. Each student will develop the ability to understand and respect people of different race; sex; ability; cultural heritage; national origin; religion; and political, economic, and social background, and their values, beliefs and attitudes.
7. Each student will acquire knowledge of the ecological sequences of choices in the use of the environment and natural resources.
8. Each student will develop general career skills, attitudes, and work habits and make a self-assessment of career prospects. Students not directly pursuing postsecondary education will acquire entry-level employment skills.
9. Each student will learn knowledge, skills, and attitudes which enable development of:
   - 9.1 Self-esteem.
   - 9.2 The ability to maintain physical, mental and emotional health.
   - 9.3 Understanding of the ill effects of alcohol, tobacco and other drugs.
10. Each student will develop a commitment to lifetime learning with the capacity for undertaking new studies, synthesizing new knowledge and experience with the known, and refining the ability to judge.

Appendix B

STUDENTS WITH HANDICAPPING CONDITIONS

The Board of Regents, through revising the Part 100 Regulations of the Commissioner and the Action Plan, has made a strong commitment to integrating the education of students with handicapping conditions into the total school program. According to Section 100.2(s) "Each student with a handicapping condition, as such term is defined in Section 200.1(ii) of this Chapter, shall have access to the full range of programs and services set forth in this Part to the extent that such programs and services are appropriate to such student's special educational needs." Districts must have policies and procedures in place to make sure that students with handicapping conditions have equal opportunities to access diploma credits, courses and requirements.

The majority of students with disabilities have the intellectual potential to master the curricula content requirements for a high school diploma. Most students who require special education attend regular education classes in conjunction with specialized instruction and/or related services. These students must attain the same academic standards as their nonhandicapped peers in order to meet these requirements. For this reason, it is very important that at all grade levels students with handicapping conditions receive instruction in the same content areas so as to receive the same informational base that will be required for proficiency on statewide testing programs and diploma requirements.

The teacher providing instruction through this syllabus/curriculum has the opportunity to provide an educational setting which will enable the students to explore their abilities and interests. Instruction may be provided to handicapped students either by teachers certified in this subject area or by special education teachers. Teachers certified in this subject area would be providing instruction to handicapped students who are recommended by the Committee on Special Education (CSE) as being able to benefit from instruction in a regular educational setting and are appropriately placed in this setting. Special education teachers may also provide this instruction to a class of students with handicapping conditions in a special class setting.

Teachers certified in the subject area should become aware of the needs of students with handicapping conditions participating in their classes. Instructional techniques and materials must be modified to the extent appropriate to provide students with handicapping conditions the opportunity to meet diploma requirements. Information or assistance is available through special education teachers, administrators, the Committee on Special Education or a student's Individualized Education Program (IEP).

1. Students with physical disabilities who use alternative communication devices may be restricted in their ability to do mathematical computations. For example, students who communicate with a head pointer and a memowriter (a calculator with adaptive large keys producing a narrow paper printout) are unable to type out the entire horizontal algorithm due to the small width of the print-out paper. Horizontal equations which do not fit on one line are broken up on several different typed lines. In addition, when typing fractions, exponents, etc., each row of digits must be typed on separate lines. Scribes may be needed to individually assist these students during notetaking and computational activities.
2. Some students learn through different modalities, thus needing visual or auditory reinforcement of information that is presented. For instance, a student with a visual impairment may require verbal descriptions of board work and text materials; a student with a hearing impairment may need a detailed visual illustration with the board work and text material; or a student with a learning disability may need memory aids such as cue cards, arrows, numbering and/or index cards with directions on them.
3. Students have many different learning styles which may affect the manner in which they acquire information. For instance, some students may learn global concepts first and then learn algorithms, whereas some students remember routines but have difficulty understanding broader applications and concepts. A

variety of presentation styles may be needed to accommodate these different learning styles.

4. Identify, define and preteach key vocabulary. Many terms in this syllabus are specific and may need continuous reinforcement for some students with handicapping conditions. It would also be helpful to provide a list of these key words to the special education teacher in order to provide additional reinforcement in the special educational setting.
5. Check periodically to determine student understanding of lectures, discussion, demonstrations, etc. and how this is related to the overall topic. Encourage students to express their understanding. It may be necessary to have small group discussions or work with a partner to determine this.
6. Provide students and special education teachers with a tape of lectures that contain substantial new vocabulary content and of guest speakers for further review within their special education class.
7. Assign a partner for the duration of a unit to a student as an additional resource to facilitate clarification of daily assignments; timelines for assignments and access to daily class notes.
8. When assigning long-term projects/reports, provide a timeline with benchmarks as indicators for completion of major project/report sections. Students who have difficulty with organizational skills and time sequence, may need to see completion of sections to maintain the organization of a lengthy project/report.

Special education teachers providing this instruction must also become familiar with the goals and objectives of the curriculum. It is important that these teachers provide their students with the same or equivalent information contained in the curriculum.

Regardless of who provides the instruction, the cooperation between teachers of regular and of special education programs is essential. It is important for the students as well as the total school environment.

Alternative Testing Techniques

Another consideration in assisting students with handicapping conditions to meet the requirements of regular education is the use of alternative testing techniques. Alternative testing techniques are modifications of testing procedures or formats which provide students with handicapping conditions equal opportunity to participate in testing situations. Such techniques provide the opportunity to demonstrate mastery of skills and attainment of knowledge without being limited or unfairly restricted by the existence of a handicapping condition.

The Committee on Special Education (CSE) is responsible for identifying and documenting a student's need for alternative testing techniques. This determination is made when a student is initially referred to CSE, is reviewed annually for as long as the student receives special education services, and is reviewed when the student is determined to no longer need special education services. **The modifications are to be used consistently throughout the student's educational program.** Principals ensure that students who have been identified by CSE as educationally handicapped are provided with the alternative testing techniques which have been recommended by CSE and approved by the Board of Education.

Alternative testing techniques which have been specified on student IEPs for use by a student must be used consistently in both special and regular education settings. Regular classroom teachers should be aware of possible alternative testing techniques and should be skilled in their implementation.

The coordination and cooperation of the total school program will assist in providing the opportunity for a greater number of students with a handicapping condition to meet the requirements needed to pursue a high school diploma. The integrated provision of regular education programs, special education programs, remediation, alternative testing techniques, modified teacher techniques and materials, and access to credit through alternatives will assist in enabling such students to pursue high school diplomas to a greater degree. The teacher who provides instruction through this curriculum has a unique opportunity to assist such students in their individual goals.

Additional information on alternative testing modifications is available in the manual entitled *Alternative Techniques for Students with Handicapping Conditions*, which can be obtained from:

New York State Education Department
Office for Education of Children with
Handicapping Conditions
Room 1071 Education Building Annex
Albany, NY 12234

Infusing Awareness of Persons with Disabilities Through Curriculum

In keeping with the concept of integration, the following subgoal of the Action Plan was established:

In all subject areas, revisions in the syllabi will include materials and activities related to generic subgoals such as problem solving, reasoning skills, speaking, capacity to search for information, the use of libraries and increasing student awareness of and *information about the disabled.*

The purpose of this subgoal is to ensure that appropriate activities and materials are available to increase student awareness of disabilities and issues in regard to disabilities.

This curriculum, by design, includes information, activities and materials regarding persons with handicapping conditions. Teachers are encouraged to include other examples as may be appropriate to their classrooms or the situation at hand. Teachers are also encouraged to assess the classroom environment to determine how the environment may contribute to student awareness of persons with disabilities.